

## STACKS OF RAMIFIED ABELIAN COVERS

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**ABSTRACT.** Given a flat, finite group scheme  $G$  finitely presented over a base scheme we introduce the notion of ramified Galois cover of group  $G$  (or simply  $G$ -cover), which generalizes the notion of  $G$ -torsor. We study the stack of  $G$ -covers, denoted with  $G\text{-Cov}$ , mainly in the abelian case, precisely when  $G$  is a finite diagonalizable group scheme over  $\mathbb{Z}$ . In this case we prove that  $G\text{-Cov}$  is connected, but it is irreducible or smooth only in few finitely many cases. On the other hand, it contains a 'special' irreducible component  $\mathcal{Z}_G$ , which is the closure of  $BG$  and this reflects the deep connection we establish between  $G\text{-Cov}$  and the equivariant Hilbert schemes. We introduce 'parametrization' maps from smooth stacks, whose objects are collections of invertible sheaves with additional data, to  $\mathcal{Z}_G$  and we establish sufficient conditions for a  $G$ -cover in order to be obtained (uniquely) through those constructions. Moreover a toric description of the smooth locus of  $\mathcal{Z}_G$  is provided.

## INTRODUCTION

In this paper we study  $G$ -Galois covers of very general schemes. Let  $G$  be a flat, finite group scheme finitely presented over a base scheme (say over a field, or, as in this paper, over  $\mathbb{Z}$ ). We define a (ramified)  $G$ -cover as a finite morphism  $f: X \rightarrow Y$  with an action of  $G$  on  $X$  such that  $f$  is  $G$ -equivariant and  $f_*\mathcal{O}_X$  is fppf-locally isomorphic to the regular representation  $\mathcal{O}_Y[G]$  as  $\mathcal{O}_Y[G]$ -comodule. This definition is somehow the most natural: it generalizes the notion of  $G$ -torsors and, under suitable hypothesis, coincides with the usual definition of Galois cover when the group  $G$  is constant (see for example [Par91, AP11, Eas08]). Moreover, as explained below, in the abelian case  $G$ -covers are strictly related to the theory of equivariant Hilbert schemes (see for example [Nak01, SP02, HS02, AB05]).

We call  $G\text{-Cov}$  the stack of  $G$ -covers and the aim of this article will be to describe its structure, especially in the abelian case. Our first result is:

**Theorem.** [1.2, 1.9] *The stack  $G\text{-Cov}$  is algebraic and finitely presented over  $S$ . Moreover  $BG$  is an open substack of  $G\text{-Cov}$ .*

In many concrete problems, one is interested in a more direct and concrete description of a  $G$ -cover  $f: X \rightarrow Y$ . This is very simple and well known when  $G = \mu_2$ : such a cover  $f$  is given by an invertible sheaf  $\mathcal{L}$  on  $Y$  with a section of  $\mathcal{L}^{\otimes 2}$ . Similarly, when  $G = \mu_3$ , a  $\mu_3$ -cover  $f$  is given by a pair  $(\mathcal{L}_1, \mathcal{L}_2)$  of invertible sheaves on  $Y$  with maps  $\mathcal{L}_1^{\otimes 2} \rightarrow \mathcal{L}_2$  and  $\mathcal{L}_2^{\otimes 2} \rightarrow \mathcal{L}_1$  (see [AV04, § 6]).

In general, however, there is no comparable description of  $G$ -covers. Very little is known when  $G$  is not abelian, beyond the cases  $G = S_d$  with  $d = 3, 4, 5$ : see [Eas08] for the case  $G = S_3$  and [Mir85, HM99, Cas96a, Cas96b] for the non-Galois case (of course, ramified covers of degree  $d$  are strictly linked with ramified  $S_d$ -covers).

Even in the abelian case, the situation become complicated very quickly when the order of  $G$  grows. The paper that inspires our work is [Par91]; here the author describes  $G$ -covers  $X \rightarrow Y$  when  $G$  is an abelian group,  $Y$  is a smooth variety over an algebraically closed field of characteristic prime to  $|G|$  and  $X$  is normal, in terms of certain invertible sheaves on  $Y$ , generalizing the description given above for  $G = \mu_2$  and  $G = \mu_3$ .

Here we concentrate on the case when  $G$  is a finite diagonalizable group scheme over  $\text{Spec } \mathbb{Z}$ ; thus,  $G$  is isomorphic to a finite direct product of group scheme of the form  $\mu_{d,\mathbb{Z}}$  for  $d \geq 1$ . We consider the dual finite abelian group  $M = \text{Hom}(G, \mathbb{G}_m)$  so that, by standard duality results (see [SGA1]),  $G$  is the fppf sheaf of homomorphisms  $M \rightarrow \mathbb{G}_m$  and a decomposition of  $M$  into a product of cyclic groups yields the decomposition of  $G$  into a product of  $\mu_d$ 's.

In this case we have an explicit description of a  $G$ -cover in terms of sequences of invertible sheaves. Indeed a  $G$ -cover over  $Y$  is of the form  $X = \text{Spec } \mathcal{A}$  where  $\mathcal{A}$  is a coherent sheaf of algebras over  $Y$  with a decomposition

$$\mathcal{A} = \bigoplus_{m \in M} \mathcal{L}_m \text{ s.t. } \mathcal{L}_0 = \mathcal{O}_Y, \mathcal{L}_m \text{ invertible and } \mathcal{L}_m \mathcal{L}_n \subseteq \mathcal{L}_{m+n} \text{ for any } m, n \in M$$

So a  $G$ -cover corresponds to a sequence of invertible sheaves  $(\mathcal{L}_m)_{m \in M}$  with maps  $\psi_{m,n}: \mathcal{L}_m \otimes \mathcal{L}_n \rightarrow \mathcal{L}_{m+n}$  satisfying certain rules and our principal aim will be to simplify the data necessary to describe such covers. For instance  $G$ -torsors correspond to sequences where all the maps  $\psi_{m,n}$  are isomorphism. Therefore, if  $G = \mu_l$ , a  $G$ -torsor is simply given by an invertible sheaf  $\mathcal{L} = \mathcal{L}_1$  and an isomorphism  $\mathcal{L}^{\otimes l} \simeq \mathcal{O}$ .

When  $G = \mu_2$  or  $G = \mu_3$  the description given above shows that the stack  $G\text{-Cov}$  is smooth, irreducible and very easy to describe. In the general case its structure turns out to be extremely intricate. For instance, as we will see,  $G\text{-Cov}$  is almost never irreducible, but has a 'special' irreducible component, called  $\mathcal{Z}_G$ , which is the scheme-theoretically closure of  $\text{B } G$ . This parallels what happens in the theory of  $M$ -equivariant Hilbert schemes (see [HS02, Remark 5.1]). It turns out that this theory and the one of  $G$ -covers is deeply connected: given an action of  $G$  on  $\mathbb{A}^r$ , induced by elements  $\underline{m} = m_1, \dots, m_r \in M$ , the equivariant Hilbert scheme  $M\text{-Hilb } \mathbb{A}^r$ , that we will denote by  $M\text{-Hilb}^{\underline{m}}$  to underline the dependency on the sequence  $\underline{m}$ , can be viewed as the functor whose objects are  $G$ -covers with an equivariant closed immersion in  $\mathbb{A}^r$ . The forgetful map  $\vartheta: M\text{-Hilb}^{\underline{m}} \rightarrow G\text{-Cov}$  is smooth and an atlas provided that  $\underline{m}$  contains all the elements in  $M - \{0\}$  (3.8). Moreover  $\vartheta^{-1}(\mathcal{Z}_G)$  coincides with the main component of  $M\text{-Hilb}^{\underline{m}}$ , first studied by Nakamura in [Nak01].

We will prove the following results on the structure of  $G\text{-Cov}$ .

**Theorem.** [3.13, 3.17, 3.18, 3.20]  *$G\text{-Cov}$  is*

- flat and of finite type over  $\mathbb{Z}$  with geometrically connected fibers,
- smooth if and only if  $G \simeq \mu_2, \mu_3, \mu_2 \times \mu_2$ ,
- normal if  $G \simeq \mu_4$ ,
- reducible if  $|G| \geq 8$  and  $G \not\simeq (\mu_2)^3$ .

*The above properties continue to hold if we replace  $G\text{-Cov}$  with  $M\text{-Hilb}^{\underline{m}}$  if  $M - \{0\} \subseteq \underline{m}$ .*

We don't know whether  $G\text{-Cov}$  is integral for  $G \simeq \mu_5, \mu_6, \mu_7, (\mu_2)^3$ . So  $G\text{-Cov}$  is usually reducible, its structure is extremely complicated and we have little hope of getting to a real understanding of the components that don't contain  $\text{B } G$ . Therefore we will focus on the main irreducible component  $\mathcal{Z}_G$  of  $G\text{-Cov}$ . The main idea behind this paper, inspired by the results in [Par91], is to try to decompose the multiplications  $\psi_{m,n} \in \mathcal{L}_{m+n} \otimes \mathcal{L}_m^{-1} \otimes \mathcal{L}_n^{-1}$  as a tensor product of sections of other invertible sheaves. Following this idea we will construct parametrization maps  $\pi_{\underline{e}}: \mathcal{F}_{\underline{e}} \rightarrow \mathcal{Z}_G \subseteq G\text{-Cov}$ , where  $\mathcal{F}_{\underline{e}}$  are 'nice' stacks, for example smooth and irreducible, whose objects are those decompositions. This construction can be better understood locally, where a  $G$ -cover over  $Y = \text{Spec } R$  is just  $X = \text{Spec } A$ , where  $A$  is an  $R$ -algebra with an  $R$ -basis  $\{v_m\}_{m \in M}$ ,  $v_0 = 1$  ( $\mathcal{L}_m = \mathcal{O}_Y v_m$ ), so that the multiplications are elements  $\psi_{m,n} \in R$  such that  $v_m v_n = \psi_{m,n} v_{m+n}$ .

Consider  $a \in R$ , a collection of natural numbers  $\mathcal{E} = (\mathcal{E}_{m,n})_{m,n \in N}$  and set  $\psi_{m,n} = a^{\mathcal{E}_{m,n}}$ . The condition that the product structure on  $A = \bigoplus_m Rv_m$  defined by the  $\psi_{m,n}$  yields an associative, commutative  $R$ -algebra, i.e. makes  $\text{Spec } A$  into a  $G$ -cover over  $\text{Spec } R$ , translates in some additive relations on the numbers  $\mathcal{E}_{m,n}$ . Call  $K_+^\vee$  the set of such collections  $\mathcal{E}$ . More generally given  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$  we can define a parametrization

$$R^r \ni (a_1, \dots, a_r) \longrightarrow \psi_{m,n} = a_1^{\mathcal{E}_{m,n}^1} \dots a_r^{\mathcal{E}_{m,n}^r}$$

This is essentially the local behavior of the map  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow G\text{-Cov}$ . In the global case the elements  $a_i$  will be sections of invertible sheaves.

From this point of view the natural questions are: given a  $G$ -cover over a scheme  $Y$  when does there exist a lift to an object of  $\mathcal{F}_{\underline{\mathcal{E}}}(Y)$ ? Is this lift unique? How can we choose the sequence  $\underline{\mathcal{E}}$ ?

The key point is to give an interpretation to  $K_+^\vee$  (that also explains this notation). Consider  $\mathbb{Z}^M$  with canonical basis  $(e_m)_{m \in M}$  and define  $v_{m,n} = e_m + e_n - e_{m+n} \in \mathbb{Z}^M / \langle e_0 \rangle$ . If  $p: \mathbb{Z}^M / \langle e_0 \rangle \longrightarrow M$  is the map  $p(e_m) = m$ , the  $v_{m,n}$  generate  $\text{Ker } p$ . Now call  $K_+$  the submonoid of  $\mathbb{Z}^M / \langle e_0 \rangle$  generated by the  $v_{m,n}$ ,  $K = \text{Ker } p$  its associated group and also consider the torus  $\mathcal{T} = \underline{\text{Hom}}(\mathbb{Z}^M / \langle e_0 \rangle, \mathbb{G}_m)$ , which acts on  $\text{Spec } \mathbb{Z}[K_+]$ . By construction we have that a collection of natural numbers  $(\mathcal{E}_{m,n})_{m,n \in M}$  belongs to  $K_+^\vee$  if and only if the association  $v_{m,n} \longrightarrow \mathcal{E}_{m,n}$  defines an additive map  $K_+ \longrightarrow \mathbb{N}$ . Therefore, as the symbol suggests, we can identify  $K_+^\vee$  with  $\text{Hom}(K_+, \mathbb{N})$ , the dual monoid of  $K_+$ . Its elements will be called *rays*. More generally a monoid map  $\psi: K_+ \longrightarrow (R, \cdot)$ , where  $R$  is a ring, yields a multiplication  $\psi_{m,n} = \psi(v_{m,n})$  and therefore we obtain a map  $\text{Spec } \mathbb{Z}[K_+] \longrightarrow \mathcal{Z}_G$ . We will prove that (see 3.6):

**Theorem.** *We have  $\mathcal{Z}_G \simeq [\text{Spec } \mathbb{Z}[K_+]/\mathcal{T}]$  and  $BG \simeq [\text{Spec } \mathbb{Z}[K]/\mathcal{T}]$ .*

Given  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$  we have defined a map  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G$ . Notice that if  $\underline{\gamma}$  is a subsequence of  $\underline{\mathcal{E}}$  then  $\mathcal{F}_{\underline{\gamma}}$  is an open substack of  $\mathcal{F}_{\underline{\mathcal{E}}}$  and  $(\pi_{\underline{\mathcal{E}}})|_{\mathcal{F}_{\underline{\gamma}}} = \pi_{\underline{\gamma}}$ . The lifting problem for the maps  $\pi_{\underline{\mathcal{E}}}$  clearly depends on the choice of the sequence  $\underline{\mathcal{E}}$ . Considering larger  $\underline{\mathcal{E}}$  allows to parametrize more covers, but also makes uniqueness of the lifting unlikely. In this direction we have proved that:

**Theorem.** [2.22] *Let  $k$  be an algebraically closed field and suppose to have a collection  $\underline{\mathcal{E}}$  whose rays generate the rational cone  $K_+^\vee \otimes \mathbb{Q}$ . Then  $\mathcal{F}_{\underline{\mathcal{E}}}(k) \longrightarrow \mathcal{Z}_G(k)$  is essentially surjective. In other words a  $G$ -cover of  $\text{Spec } k$  in the main component  $\mathcal{Z}_G$  has a multiplication of the form  $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$  for some  $\mathcal{E} \in K_+^\vee$ .*

On the other hand small sequences  $\underline{\mathcal{E}}$  can guarantee uniqueness but not existence. The solution we have found is to consider a particular class of rays, called extremal, that have minimal non empty support. Set  $\underline{\eta}$  for the sequence of all extremal rays (that is finite). Notice that extremal rays generate  $K_+^\vee \otimes \mathbb{Q}$ . We prove that:

**Theorem.** [2.47, 2.48] *The smooth locus  $\mathcal{Z}_G^{\text{sm}}$  of  $\mathcal{Z}_G$  is of the form  $[X_G/\mathcal{T}]$  where  $X_G$  is a smooth toric variety over  $\mathbb{Z}$  (whose maximal torus is  $\text{Spec } \mathbb{Z}[K]$ ). Moreover  $\pi_{\underline{\eta}}: \mathcal{F}_{\underline{\eta}} \longrightarrow \mathcal{Z}_G$  induces an isomorphism of stacks*

$$\pi_{\underline{\eta}}^{-1}(\mathcal{Z}_G^{\text{sm}}) \xrightarrow{\simeq} \mathcal{Z}_G^{\text{sm}}$$

Among the extremal rays there are special rays, called smooth, that can be defined as extremal rays  $\mathcal{E}$  whose associated multiplication  $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$  yields a cover in  $\mathcal{Z}_G^{\text{sm}}$ . Set  $\underline{\xi}$  for the sequence of smooth extremal rays. It turns out that theorem above holds if we replace  $\underline{\eta}$  with  $\underline{\xi}$ .

If we set  $\underline{\text{Pic}} X$  for the category of invertible sheaves on  $X$  and any map we also have:

**Theorem.** [2.52] Consider a 2-commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{F}_{\underline{\mathcal{E}}} \\ f \downarrow & \nearrow \text{dashed} & \downarrow \pi_{\underline{\mathcal{E}}} \\ X & \longrightarrow & G\text{-Cov} \end{array}$$

where  $X, Y$  are schemes and  $\underline{\mathcal{E}}$  is a sequence of elements of  $K_+^\vee$ . If  $\underline{\text{Pic}} X \xrightarrow{f^*} \underline{\text{Pic}} Y$  is fully faithful (an equivalence) the dashed lifting is unique (exists).

In particular theorems above allow to conclude that:

**Theorem.** [2.48, 2.53] Let  $X$  be a locally noetherian and locally factorial scheme. A cover  $\chi \in G\text{-Cov}(X)$  such that  $\chi|_{k(p)} \in \mathcal{Z}_G^{\text{sm}}(k(p))$  for any  $p \in X$  with  $\text{codim}_p X \leq 1$  lifts uniquely to  $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ .

An interesting problem is to describe all (smooth) extremal rays. This seems very difficult and it is related to the problem of finding  $\mathbb{Q}$ -linearly independent sequences among the  $v_{m,n} \in K_+$ . A natural way of obtaining extremal rays is trying to describe  $G$ -covers with special properties. The first examples of them arise looking at covers with normal total space. Indeed in [Par91] the author is able to describe the multiplications yielding regular  $G$ -covers of a DVR. This description, using the language introduced above, yields a sequence  $\underline{\delta} = (\mathcal{E}^\phi)_{\phi \in \Phi_M}$  of smooth extremal rays, where  $\Phi_M$  is the set of surjective maps  $M \rightarrow \mathbb{Z}/d\mathbb{Z}$  with  $d > 1$ . In this paper we will define a stratification of  $G\text{-Cov}$  by open substacks  $BG = U_0 \subseteq U_1 \subseteq \dots \subseteq U_{|G|-1} = G\text{-Cov}$  and we will prove that there exists an explicitly given sequence  $\underline{\mathcal{E}}$  of smooth integral extremal rays (defined in 4.39) containing  $\underline{\delta}$  such that:

**Theorem.** [3.40, 4.41] We have that  $U_2 \subseteq \mathcal{Z}_G^{\text{sm}}$  and that  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{Z}_G$  induces isomorphisms of stacks

$$\pi_{\underline{\mathcal{E}}}^{-1}(U_2) \xrightarrow{\simeq} U_2, \quad \pi_{\underline{\delta}}^{-1}(U_1) = \pi_{\underline{\mathcal{E}}}^{-1}(U_1) \xrightarrow{\simeq} U_1$$

Theorem above implies that  $\text{M-Hilb } \mathbb{A}^2$  is smooth and irreducible (4.42). In this way we get an alternative proof of the result in [Mac03] (later generalized in [MS10]) in the particular case of equivariant Hilbert schemes.

**Theorem.** [3.41, 4.44] Let  $X$  be a locally noetherian and locally factorial scheme and  $\chi \in G\text{-Cov}(X)$ . If  $\chi|_{k(p)} \in U_1$  for any  $p \in X$  with  $\text{codim}_p X \leq 1$ , then  $\chi$  lifts uniquely to  $\mathcal{F}_{\underline{\delta}}(X)$ . If  $\chi|_{k(p)} \in U_2$  for any  $p \in X$  with  $\text{codim}_p X \leq 1$ , then  $\chi$  lifts uniquely to  $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ .

Notice that  $\underline{\mathcal{E}} = \underline{\delta}$  if and only if  $G \simeq (\mu_2)^l$  or  $G \simeq (\mu_3)^l$  (4.43). Finally we prove:

**Theorem.** [3.42, 4.54] Let  $X$  be a locally noetherian and locally factorial integral scheme with  $\dim X \geq 1$  and  $(\text{char } X, |M|) = 1$  and  $Y/X$  be a  $G$ -cover. If  $Y$  is regular in codimension 1 it is normal and  $Y/X$  comes from a unique object of  $\mathcal{F}_{\underline{\delta}}(X)$ . If  $Y$  is normal crossing in codimension 1 (see 4.46) then  $Y/X$  comes from a unique object of  $\mathcal{F}_{\underline{\gamma}}(X)$ , where  $\underline{\delta} \subseteq \underline{\gamma} \subseteq \underline{\mathcal{E}}$  is an explicitly given sequence.

The part concerning regular in codimension 1 covers is essentially a rewriting of Theorem 2.1 and Corollary 3.1 of [Par91] extended to locally noetherian and locally factorial schemes, while the last part generalizes Theorem 1.9 of [AP11].

**Table of contents.** We now briefly summarize how this paper is divided.

*Section 1.* We introduce the notion of  $G$ -covers and prove some general facts about them, e.g. the algebraicity of  $G\text{-Cov}$ .

All the other sections will be dedicated to the study of  $G$ -Cov when  $G$  is a finite diagonalizable group with dual group  $M = \text{Hom}(G, \mathbb{G}_m)$ .

*Section 2.*  $G$ -Cov and some of its substacks, like  $\mathcal{Z}_G$  and  $\text{B } G$ , share a common structure, i.e. they are all of the form  $\mathcal{X}_\phi = [\text{Spec } \mathbb{Z}[T_+]/\mathcal{T}]$ , where  $T_+$  is a finitely generated commutative monoid whose associated group is free of finite rank,  $\mathcal{T}$  is a torus over  $\mathbb{Z}$  and  $\phi: T_+ \rightarrow \mathbb{Z}^r$  is an additive map, that induces the action of  $\mathcal{T}$  on  $\text{Spec } \mathbb{Z}[T_+]$ . Section 2 will be dedicated to the study of such stacks. As we will see many facts about  $G$ -Cov are just applications of general results about such stacks. For instance the existence of a special irreducible component  $\mathcal{Z}_\phi$  of  $\mathcal{X}_\phi$  as well as the use of  $T_+^\vee = \text{Hom}(T_+, \mathbb{N})$  for the study of the smooth locus of  $\mathcal{Z}_\phi$  are properties that can be stated in this setting.

*Section 3.* We will explain how  $G$ -Cov can be viewed as a stack of the form  $\mathcal{X}_\phi$  and how it is related to the equivariant Hilbert schemes. Then we will study the properties of connectedness, irreducibility and smoothness for  $G$ -Cov. Finally we will introduce the stratification  $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|G|-1} = G\text{-Cov}$  and we will characterize the locus  $U_1$ .

*Section 4.* We will study the locus  $U_2$  and  $G$ -covers whose total space is normal crossing in codimension 1.

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## NOTATIONS

A map of schemes  $f: X \rightarrow Y$  will be called a *cover* if it is finite, flat and of finite presentation or, equivalently, if it is affine and  $f_*\mathcal{O}_X$  is locally free of finite rank. If  $X$  is a scheme and  $p \in X$  we set  $\text{codim}_p X = \dim \mathcal{O}_{X,p}$  and we will denote with  $X^{(1)} = \{p \in X \mid \text{codim}_p X = 1\}$  the set of codimension 1 points of  $X$ .

If  $N$  is an abelian group we set  $D(N) = \underline{\text{Hom}}_{\text{groups}}(N, \mathbb{G}_m)$  for the diagonalizable group associated to it, while if  $f: G \rightarrow S$  is an affine group scheme we set  $\mathcal{O}_S[G] = f_*\mathcal{O}_G$ .

Given an element  $f = \sum_i a_i e_i \in \mathbb{Z}^r$  and invertible sheaves  $\mathcal{L}_1, \dots, \mathcal{L}_r$  on a scheme we will use the notation

$$\underline{\mathcal{L}}^f = \bigotimes_i \mathcal{L}_i^{\otimes a_i}, \quad \text{Sym}^* \underline{\mathcal{L}} = \text{Sym}^*(\mathcal{L}_1, \dots, \mathcal{L}_r) = \bigoplus_{g \in \mathbb{Z}^r} \underline{\mathcal{L}}^g$$

Notice also that, if for any  $i$  we have  $\mathcal{L}_i = \mathcal{O}$ , then there is a canonical isomorphism  $\underline{\mathcal{L}}^f \simeq \mathcal{O}$ .

Finally if  $\mathcal{X}$  is an algebraic stack we denote with  $|\mathcal{X}|$  its associated topological space.

## 1. $G$ -COVERS

In this section we will fix a base scheme  $S$  and a group scheme  $G$  over it, which is also a cover of  $S$ . We will denote by  $\mathcal{A}$  the regular representation of  $G$ , i.e.  $\mathcal{A} = \mathcal{O}_S[G]$  with the  $G$ -comodule structure  $\mu: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{O}_S[G]$  induced by the multiplication of  $G$ .

The aim of this section is to introduce the notion of a ramified Galois cover and prove that the associated stack is algebraic.

**Definition 1.1.** Given a scheme  $T$  over  $S$ , a *ramified Galois cover of group  $G$* , or simply a  *$G$ -cover*, over it is a cover  $X \xrightarrow{f} T$  together with an action of  $G_T$  on it such that there exists an fppf coverings  $\{U_i \rightarrow T\}$  and isomorphisms of  $G$ -comodules

$$(f_* \mathcal{O}_X)|_{U_i} \simeq \mathcal{A}|_{U_i}$$

We will call  $G\text{-Cov}(T)$  the groupoid of  $G$ -covers over  $T$ .

The  $G$ -covers form a stack  $G\text{-Cov}$  over  $S$ . Moreover any  $G$ -torsor is a  $G$ -cover and more precisely we have:

**Proposition 1.2.**  $BG$  is an open substack of  $G\text{-Cov}$ .

*Proof.* Given a scheme  $U$  over  $S$  and a  $G$ -cover  $X = \text{Spec } \mathcal{B}$  over  $U$ ,  $X$  is a  $G$ -torsor if and only if the map  $G \times X \rightarrow X \times X$  is an isomorphism. This map is induced by a map  $\mathcal{B} \otimes \mathcal{B} \xrightarrow{h} \mathcal{B} \otimes \mathcal{O}[G_U]$  and so the locus over which  $X$  is a  $G$ -torsor is given by the vanishing of  $\text{Coker } h$ , which is an open subset.  $\square$

**Definition 1.3.** The main component  $\mathcal{Z}_G$  of  $G\text{-Cov}$  is the reduced closed substack induced by the closure of  $BG$  in  $G\text{-Cov}$ .

In order to prove that  $G\text{-Cov}$  is an algebraic stack we will present it as a quotient stack by a smooth group scheme.

*Notation 1.4.* Let  $S$  be a scheme and  $\mathcal{F} \in \text{QCoh } S$ . We define  $W(\mathcal{F}): (\text{Sch}/S)^{\text{op}} \rightarrow \text{set}$  as

$$W(\mathcal{F})(U \xrightarrow{f} S) = H^0(U, f^* \mathcal{F})$$

Remember that if  $\mathcal{F}$  is a locally free sheaf of finite rank, then  $W(\mathcal{F})$  is smooth, affine and finitely presented over  $S$ .

**Proposition 1.5.** *The functor*

$$\begin{aligned} (\text{Sch}/S)^{\text{op}} &\xrightarrow{X_G} \text{set} \\ T &\longmapsto \{\mathcal{O}[G_T]\text{-coalgebra structures on } \mathcal{A}_T\} \end{aligned}$$

*is an affine scheme finitely presented over  $S$ .*

*Proof.* Denote by  $m_G: \mathcal{O}[G] \otimes \mathcal{O}[G] \rightarrow \mathcal{O}[G]$  the multiplication map of  $\mathcal{O}[G]$ . Let also  $T$  be a scheme over  $S$ . An element of  $X_G(T)$  is given by maps

$$\mathcal{A}_T \otimes \mathcal{A}_T \xrightarrow{m} \mathcal{A}_T, \quad \mathcal{O}_T \xrightarrow{e} \mathcal{A}_T$$

for which  $\mathcal{A}$  becomes a sheaf of algebras with multiplication  $m$  and identity  $e(1)$  and such that  $\mu$  is a homomorphism of algebras over  $\mathcal{O}_T$ . In particular  $e$  has to be an isomorphism onto  $\mathcal{A}^G = \mathcal{O}_T$ . Therefore we have an inclusion  $X_G \subseteq \underline{\text{Hom}}(W(\mathcal{A} \otimes \mathcal{A}), W(\mathcal{A})) \times \mathbb{G}_m$  which is locally scheme-theoretically defined by the vanishing of certain polynomials.  $\square$

**Proposition 1.6.**  $\underline{\text{Aut}}^G W(\mathcal{A})$  is a smooth group scheme finitely presented over  $S$ .

*Proof.* The morphisms

$$\begin{aligned} \varepsilon \circ \phi &\longleftarrow \phi \\ \mathcal{O}_S[G]^\vee &\longrightarrow \underline{\text{End}}^G \mathcal{A} \\ f &\longmapsto (f \otimes \text{id}) \circ \Delta \end{aligned}$$

where  $\Delta$  and  $\varepsilon$  are respectively the co-multiplication and the co-unity of  $\mathcal{O}_S[G]$ , are one the inverse of the other. In particular we obtain an isomorphism  $\underline{\text{End}}^G W(\mathcal{A}) \simeq W(\mathcal{O}_S[G]^\vee)$ , so that  $\underline{\text{End}}^G W(\mathcal{A})$  and its open subscheme  $\underline{\text{Aut}}^G W(\mathcal{A})$  are smooth and finitely presented over  $S$ .  $\square$

*Remark 1.7.*  $\underline{\text{Aut}}^G W(\mathcal{A})$  acts on  $X_G$  in the following way. Given a scheme  $T$  over  $S$ , a  $G$ -equivariant automorphism  $f: \mathcal{A}_T \rightarrow \mathcal{A}_T$  and  $(m, e) \in X_G(T)$  we can set  $f(m, e)$  for the unique structure of sheaf of algebras on  $\mathcal{A}_T$  such that  $f: (\mathcal{A}_T, m, e) \rightarrow (\mathcal{A}_T, f(m, e))$  is an isomorphism of  $\mathcal{O}_T$ -algebras.

**Proposition 1.8.** *The natural map  $X_G \xrightarrow{\pi} G\text{-Cov}$  is an  $\underline{\text{Aut}}^G W(\mathcal{A})$ -torsor, i.e.*

$$G\text{-Cov} \simeq [X_G / \underline{\text{Aut}}^G W(\mathcal{A})]$$

*Proof.* Consider a cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & X_G \\ \downarrow & & \downarrow \pi \\ U & \xrightarrow{f} & G\text{-Cov} \end{array}$$

where  $U$  is a scheme and  $f: Y \rightarrow U$  is a  $G$ -cover. We want to prove that  $P$  is an  $\underline{\text{Aut}}^G W(\mathcal{A})$  torsor over  $U$  and that the map  $P \rightarrow X_G$  is equivariant. Since  $\pi$  is an fppf epimorphism, we can assume that  $f$  comes from  $X_G$ , i.e.  $f_* \mathcal{O}_Y = \mathcal{A}_U$  with multiplication  $m$  and neutral element  $e$ . It is now easy to prove that

$$\begin{array}{ccc} \underline{\text{Aut}}^G W(\mathcal{A}_U) & \xrightarrow{\simeq} & P \\ h & \longmapsto & h(m, e) \end{array}$$

is a bijection and that all the other claims hold.  $\square$

Using above propositions we can conclude that:

**Theorem 1.9.** *The stack  $G\text{-Cov}$  is algebraic and finitely presented over  $S$ .*

## 2. THE STACK $\mathcal{X}_\phi$ .

In the following sections we will study the stack  $G\text{-Cov}$  when  $G = D(M)$ , the diagonalizable group of a finite abelian group  $M$ . The structure of this stack and of some of its substacks is in somehow special and in this section we will provide general constructions and properties that will be used later. Given a monoid map  $T_+ \xrightarrow{\phi} \mathbb{Z}^r$ , we will associate a stack  $\mathcal{X}_\phi$  whose objects are sequences of invertible sheaves with additional data and we will study particular 'parametrization' of these objects, defined by a map of stack  $\mathcal{F}_\mathcal{E} \xrightarrow{\pi_\mathcal{E}} \mathcal{X}_\phi$ , where  $\mathcal{F}_\mathcal{E}$  will be a 'nice' stack.

In this section we will consider given a commutative monoid  $T_+$  together to a monoid map  $\phi: T_+ \rightarrow \mathbb{Z}^r$ .

**Definition 2.1.** We define the stack  $\mathcal{X}_\phi$  over  $\mathbb{Z}$  as follows.

- *Objects.* An object over a scheme  $S$  is a pair  $(\underline{\mathcal{L}}, a)$  where:
  - $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$  are invertible sheaves on  $S$ ;
  - $T_+ \xrightarrow{a} \text{Sym}^* \underline{\mathcal{L}}$  is an additive map such that  $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$  for any  $t \in T_+$ .
- *Arrows.* An isomorphism  $(\underline{\mathcal{L}}, a) \xrightarrow{\sigma} (\underline{\mathcal{L}}', a')$  of objects over  $S$  is given by a sequence  $\underline{\sigma} = \sigma_1, \dots, \sigma_r$  of isomorphisms  $\sigma_i: \mathcal{L}_i \xrightarrow{\simeq} \mathcal{L}'_i$  such that

$$\underline{\sigma}^{\phi(t)}(a(t)) = a'(t) \text{ for any } t \in T_+$$

**Example 2.2.** Let  $f_1, \dots, f_s, g_1, \dots, g_t \in \mathbb{Z}^r$  and consider the stack  $\mathcal{X}_{\underline{f}, \underline{g}}$  of invertible sheaves  $\mathcal{L}_1, \dots, \mathcal{L}_r$  with maps  $\mathcal{O} \rightarrow \underline{\mathcal{L}}^{f_i}$  and  $\mathcal{O} \xrightarrow{\sim} \underline{\mathcal{L}}^{g_j}$ . If  $T_+ = \mathbb{N}^s \times \mathbb{Z}^t$  and  $\phi: T_+ \rightarrow \mathbb{Z}^r$  is the map given by the matrix  $(f_1 | \dots | f_s | g_1 | \dots | g_t)$  then  $\mathcal{X}_{\underline{f}, \underline{g}} = \mathcal{X}_\phi$ .

*Notation 2.3.* We set

$$\mathbb{Z}[T_+] = \mathbb{Z}[x_t]_{t \in T_+} / (x_t x_{t'} - x_{t+t'}, x_0 - 1)$$

and  $\mathcal{O}_S[T_+] = \mathbb{Z}[T_+] \otimes_{\mathbb{Z}} \mathcal{O}_S$ . The scheme  $\text{Spec } \mathcal{O}_S[T_+]$  over  $S$  represents the functor that associates to any scheme  $U/S$  the set of additive map  $T_+ \rightarrow (\mathcal{O}_U, *)$ .  $D(\mathbb{Z}^r)$  acts on  $\text{Spec } \mathbb{Z}[T_+]$  by the graduation  $\deg t = \phi(t)$ .

**Proposition 2.4.** Set  $X = \text{Spec } \mathbb{Z}[T_+]$ . The choice  $\mathcal{L}_i = \mathcal{O}_X$  and

$$\begin{aligned} \underline{\mathcal{L}}^{\phi(t)} &\xrightarrow{\sim} \mathcal{O}_X \\ a(t) &\longleftarrow x_t \end{aligned}$$

induces a smooth epimorphism  $X \rightarrow \mathcal{X}_\phi$  such that  $\mathcal{X}_\phi \simeq [X/D(\mathbb{Z}^r)]$ . In particular  $\mathcal{X}_\phi$  is an algebraic stack of finite type over  $\mathbb{Z}$ .

*Proof.* It's enough to note that an object of  $[X/D(\mathbb{Z}^r)](U)$  is given by invertible sheaves  $\mathcal{L}_1, \dots, \mathcal{L}_r$  with a  $D(\mathbb{Z}^r)$ -equivariant map  $\text{Spec } \text{Sym}^* \underline{\mathcal{L}} \rightarrow \text{Spec } \mathbb{Z}[T_+]$  which exactly corresponds to an additive map  $T_+ \rightarrow \text{Sym}^* \underline{\mathcal{L}}$  as in the definition of  $\mathcal{X}_\phi$ . It's easy to check that the map  $X \rightarrow [X/D(\mathbb{Z}^r)] \rightarrow \mathcal{X}_\phi$  is the same defined in the statement.  $\square$

*Remark 2.5.* Given a map  $U \xrightarrow{a} X = \text{Spec } \mathbb{Z}[T_+]$ , i.e. a monoid map  $T_+ \xrightarrow{a} \mathcal{O}_U$ , the induced object  $U \xrightarrow{a} X \rightarrow \mathcal{X}_\phi$  is the pair  $(\underline{\mathcal{L}}, \tilde{a})$  where  $\mathcal{L}_i = \mathcal{O}_U$  and for any  $t \in T_+$

$$\begin{aligned} \mathcal{O}_U &\xrightarrow{\sim} \underline{\mathcal{L}}^{\phi(t)} \\ a(t) &\longmapsto \tilde{a}(t) \end{aligned}$$

We will denote by  $a$  also the object  $(\underline{\mathcal{L}}, \tilde{a}) \in \mathcal{X}_\phi(U)$ .

Given two elements  $a, b: T_+ \rightarrow \mathcal{O}_U \in \mathcal{X}_\phi(U)$  we have

$$\text{Iso}_U(a, b) = \{\sigma_1, \dots, \sigma_r \in \mathcal{O}_U^* \mid \underline{\sigma}^{\phi(t)} a(t) = b(t) \ \forall t \in T_+\}$$

**Lemma 2.6.** Consider a commutative diagram

$$\begin{array}{ccc} T_+ & \xrightarrow{h} & T'_+ \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{Z}^r & \xrightarrow{g} & \mathbb{Z}^s \end{array}$$

where  $T_+, T'_+$  are commutative monoids and  $\phi, \psi, h, g$  are additive maps. Then we have a 2-commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbb{Z}[T'_+] & \xrightarrow{h^*} & \text{Spec } \mathbb{Z}[T_+] \\ \downarrow & & \downarrow \\ \mathcal{X}_\psi & \xrightarrow{\Lambda} & \mathcal{X}_\phi \end{array}$$

$$(2.1) \quad (\underline{\mathcal{L}}, T'_+ \xrightarrow{a} \text{Sym}^* \underline{\mathcal{L}}) \mapsto (\underline{\mathcal{M}}, T_+ \xrightarrow{b} \text{Sym}^* \underline{\mathcal{M}})$$

where, for  $i = 1, \dots, r$ ,  $\mathcal{M}_i = \underline{\mathcal{L}}^{g(e_i)}$  and  $b$  is the unique map such that



$$\begin{array}{ccccc}
T_+ & \xrightarrow{b} & \mathrm{Sym}^* \underline{\mathcal{M}} & \underline{\mathcal{M}}^v \simeq & \underline{\mathcal{L}}^{g(v)} \\
h \downarrow & & \downarrow & \downarrow \swarrow \mathrm{id} & \\
T'_+ & \xrightarrow{a} & \mathrm{Sym}^* \underline{\mathcal{L}} & \underline{\mathcal{L}}^{g(v)} & 
\end{array}$$

*Proof.* An easy computation shows that there is a canonical isomorphism  $\mathcal{M}^v \simeq \underline{\mathcal{L}}^{g(v)}$  and so  $b(t)$  corresponds under this isomorphism to  $a(h(t)) \in \underline{\mathcal{L}}^{\psi(h(t))} = \underline{\mathcal{L}}^{g(\phi(t))} \simeq \underline{\mathcal{M}}^{\phi(t)}$ . So the functor  $\Lambda$  is well defined and we have only to check the commutativity of the second diagram in the sentence. The map  $\mathrm{Spec} \mathbb{Z}[T'_+] \rightarrow \mathrm{Spec} \mathbb{Z}[T_+] \rightarrow \mathcal{X}_\phi$  is given by trivial invertible sheaves and the additive map

$$\begin{array}{ccc}
T_+ & \rightarrow & \mathbb{Z}[T_+][x_1, \dots, x_r]_{\prod_i x_i} \rightarrow \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} \\
t & \mapsto & x_t x^{\phi(t)} \mapsto x_{h(t)} x^{\phi(t)}
\end{array}$$

Instead the map  $\mathrm{Spec} \mathbb{Z}[T'_+] \rightarrow \mathcal{X}_\psi \rightarrow \mathcal{X}_\phi$  is given by trivial invertible sheaves and the map  $b$  that makes the following diagram commutative

$$\begin{array}{ccccc}
T_+ & \xrightarrow{b} & \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} & x^v & \\
h \downarrow & & \downarrow & \downarrow & \\
T'_+ & \xrightarrow{a} & \mathbb{Z}[T'_+][y_1, \dots, y_s]_{\prod_i y_i} & y^{g(v)} & \\
t & \mapsto & x_t y^{\psi(t)} & & 
\end{array}$$

Since  $x_{h(t)} x^{\phi(t)}$  is sent to  $x_{h(t)} y^{g(\phi(t))} = x_{h(t)} y^{\psi(h(t))} = a(h(t))$  we find again  $b(t) = x_{h(t)} x^{\phi(t)}$ .  $\square$

*Remark 2.7.* The functor  $\mathcal{X}_\psi \rightarrow \mathcal{X}_\phi$  sends an element  $a: T'_+ \rightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$  to the element  $a \circ h \in \mathcal{X}_\phi(U)$ . Moreover, taking into account the description given in 2.5, if  $a, b: T'_+ \rightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$  we have

$$\begin{array}{ccc}
\mathrm{Iso}_U(a, b) & \longrightarrow & \mathrm{Iso}_U(a \circ h, b \circ h) \\
\underline{\sigma} & \longmapsto & \underline{\sigma}^{g(e_1)}, \dots, \underline{\sigma}^{g(e_r)}
\end{array}$$

## 2.1. The main irreducible component $\mathcal{Z}_\phi$ of $\mathcal{X}_\phi$ .

*Notation 2.8.* A monoid will be called integral if it satisfies the cancellative rule, i.e.

$$\forall a, b, c \ a + b = a + c \implies b = c$$

We will call  $T$  the associated group and  $T_+^{int} = \mathrm{Im}(T_+ \rightarrow T)$  the associated integral monoid of  $T_+$ . This means that any monoid map  $T_+ \rightarrow S_+$ , where  $S_+$  is a group (integral monoid), factors uniquely through  $T(T_+^{int})$ .

From now on  $T_+$  will be a finitely generated monoid whose associated group is a free  $\mathbb{Z}$ -module of finite rank. We will also call  $\phi$  the induced map  $T \rightarrow \mathbb{Z}^r$ , but with the convention that the stack  $\mathcal{X}_\phi$  will always refer to the map  $T_+ \rightarrow \mathbb{Z}^r$ .

*Remark 2.9.* If  $D$  is a domain, then  $\mathrm{Spec} D[T]$  is an open subscheme of  $\mathrm{Spec} D[T_+]$ , while  $\mathrm{Spec} D[T_+^{int}]$  is an irreducible component of it. In particular we have

**Proposition 2.10.** *Let  $\hat{\phi}: T \rightarrow \mathbb{Z}^r$  be the extension of  $\phi$  and set  $\hat{\phi}^{int} = \hat{\phi}|_{T_+^{int}}$ . Then  $\mathcal{B}_\phi = \mathcal{X}_{\hat{\phi}} \rightarrow \mathcal{X}_\phi$  is an open immersion, while  $\mathcal{Z}_\phi = \mathcal{X}_{\hat{\phi}^{int}} \rightarrow \mathcal{X}_\phi$  is a closed*

one. Moreover  $\mathcal{Z}_\phi$  is the reduced closed stack associated to the closure of  $\mathcal{B}_\phi$ , it is an irreducible component of  $\mathcal{X}_\phi$  and

$$\mathcal{B}_\phi \simeq [\mathrm{Spec} \mathbb{Z}[T]/D(\mathbb{Z}^r)] \text{ and } \mathcal{Z}_\phi \simeq [\mathrm{Spec} \mathbb{Z}[T_+^{int}]/D(\mathbb{Z}^r)]$$

**Definition 2.11.** With notation above we will call respectively  $\mathcal{B}_\phi$  and  $\mathcal{Z}_\phi$  the *principal open substack* and the *main irreducible component* of  $\mathcal{X}_\phi$ .

*Notation 2.12.* We set

$$T_+^\vee = \mathrm{Hom}(T_+, \mathbb{N}) = \{\mathcal{E} \in T^* \mid \mathcal{E}(T_+) \subseteq \mathbb{N}\}$$

and we will call its elements the *integral rays* for  $T_+$ , or simply rays. Note that  $T_+^\vee = T_+^{int\vee}$ . Given  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$  we will denote by  $\underline{\mathcal{E}}$  also the induced map  $T \longrightarrow \mathbb{Z}^s$ . Moreover we set

$$\mathrm{Supp} \underline{\mathcal{E}} = \{v \in T_+ \mid \exists i \mathcal{E}^i(v) > 0\}$$

**Definition 2.13.** Given a sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$  set

$$\begin{array}{ccc} \mathbb{N}^s \oplus T & \xrightarrow{\sigma_{\underline{\mathcal{E}}}} & \mathbb{Z}^s \oplus \mathbb{Z}^r \\ e_i \mid & \longrightarrow & e_i \\ t \mid & \longrightarrow & (\underline{\mathcal{E}}(t), -\phi(t)) \end{array}$$

where  $e_1, \dots, e_s$  is the canonical basis of  $\mathbb{Z}^s$ . We will call  $\mathcal{F}_{\underline{\mathcal{E}}} = \mathcal{X}_{\sigma_{\underline{\mathcal{E}}}}$ .

*Remark 2.14.* An object of  $\mathcal{F}_{\underline{\mathcal{E}}}$  over a scheme  $U$  is given by a sequence  $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$  where:

- $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$  and  $\underline{\mathcal{M}} = (\mathcal{M}_{\mathcal{E}})_{\mathcal{E} \in \underline{\mathcal{E}}} = \mathcal{M}_1, \dots, \mathcal{M}_s$  are invertible sheaves on  $U$ ;
- $\underline{z} = (z_{\mathcal{E}})_{\mathcal{E} \in \underline{\mathcal{E}}} = z_1, \dots, z_s$  are sections  $z_i \in \mathcal{M}_i$ ;
- for any  $t \in T$ ,  $\lambda(t) = \lambda_t$  is an isomorphism  $\underline{\mathcal{L}}^{\phi(t)} \xrightarrow{\sim} \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)}$  additive in  $t$ .

An isomorphism  $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \longrightarrow (\underline{\mathcal{L}}', \underline{\mathcal{M}}', \underline{z}', \lambda')$  is a pair  $(\underline{\omega}, \underline{\tau})$  where  $\underline{\omega} = \omega_1, \dots, \omega_r$ ,  $\underline{\tau} = \tau_1, \dots, \tau_s$  are sequences of isomorphisms  $\mathcal{L}_i \xrightarrow{\omega_i} \mathcal{L}'_i$ ,  $\mathcal{M}_j \xrightarrow{\tau_j} \mathcal{M}'_j$  such that  $\tau_j(z_j) = z'_j$  and for any  $t \in T$  we have a commutative diagram

$$\begin{array}{ccc} \underline{\mathcal{L}}^{\phi(t)} & \xrightarrow{\lambda_t} & \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)} \\ \underline{\omega}^{\phi(t)} \downarrow & & \downarrow \underline{\tau}^{\phi(t)} \\ \underline{\mathcal{L}}'^{\phi(t)} & \xrightarrow{\lambda'_t} & \underline{\mathcal{M}}'^{\underline{\mathcal{E}}(t)} \end{array}$$

An object over  $U$  coming from the atlas  $\mathrm{Spec} \mathbb{Z}[\mathbb{N}^s \oplus T]$  is a pair  $(\underline{z}, \lambda)$  where  $\underline{z} = z_1, \dots, z_s \in \mathcal{O}_U$  and  $\lambda: T \longrightarrow \mathcal{O}_U^*$  is a group homomorphism. Given  $(\underline{z}, \lambda), (\underline{z}', \lambda') \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$  we have

$$\mathrm{Iso}_U((\underline{z}, \lambda), (\underline{z}', \lambda')) = \{(\underline{\omega}, \underline{\tau}) \in (\mathcal{O}_U^*)^r \times (\mathcal{O}_U^*)^s \mid \tau_i z_i = z'_i, \underline{\tau}^{\underline{\mathcal{E}}(t)} \lambda(t) = \underline{\omega}^{\phi(t)} \lambda'(t)\}$$

**Definition 2.15.** Given a sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s$  of elements of  $T_+^\vee$  we define the map

$$\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$$

induced by the commutative diagram

$$\begin{array}{ccccc} & & T_+ & \xrightarrow{\phi} & \mathbb{Z}^r \\ & & \downarrow & & \downarrow \\ t \mid & & \downarrow & & \downarrow \\ (\underline{\mathcal{E}}(t), -t) & \mathbb{N}^s \oplus T & \xrightarrow{\sigma_{\underline{\mathcal{E}}}} & \mathbb{Z}^s \oplus \mathbb{Z}^r \end{array}$$

*Remark 2.16.* We can describe the functor  $\pi_{\underline{\mathcal{E}}}$  explicitly. So suppose to have an object  $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$ . We have  $\pi_{\underline{\mathcal{E}}}(\chi) = (\underline{\mathcal{L}}, a) \in \mathcal{X}_{\phi}(U)$  where  $a$  is given, for any  $t \in T_+$ , by

$$\begin{aligned} \underline{\mathcal{L}}^{\phi(t)} &\xrightarrow{\lambda_t} \underline{\mathcal{M}}^{\mathcal{E}(t)} \\ a(t) &\longmapsto \underline{z}^{\mathcal{E}(t)} \end{aligned}$$

Instead, if  $(\underline{\omega}, \underline{\tau})$  is an isomorphism in  $\mathcal{F}_{\underline{\mathcal{E}}}$ , then  $\pi_{\underline{\mathcal{E}}}(\underline{\omega}, \underline{\tau}) = \underline{\omega}$ .

If  $(\underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$  then  $a = \pi_{\underline{\mathcal{E}}}(\underline{z}, \lambda) \in \mathcal{X}_{\phi}(U)$  is given by

$$\begin{aligned} T_+ &\longrightarrow \mathcal{O}_U \\ t &\longmapsto \underline{z}^{\mathcal{E}(t)} / \lambda_t = z_1^{\mathcal{E}^1(t)} \cdots z_s^{\mathcal{E}^s(t)} / \lambda_t \end{aligned}$$

*Remark 2.17.* If  $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$  is a sequence of elements of  $T_+^{\vee}$ ,  $J \subseteq I$  and we set  $\underline{\mathcal{E}} = (\mathcal{E}^j)_{j \in J}$  we can define a map over  $\mathcal{X}_{\phi}$  as

$$\begin{aligned} \mathcal{F}_{\underline{\mathcal{E}}} &\xrightarrow{\rho} \mathcal{F}_{\underline{\mathcal{E}}} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) &\longmapsto (\underline{\mathcal{L}}, \underline{\mathcal{M}}', \underline{z}', \lambda) \end{aligned} \quad \mathcal{M}'_i = \begin{cases} \mathcal{M}_i & i \in J \\ \mathcal{O} & i \notin J \end{cases} \quad z'_i = \begin{cases} z_i & i \in J \\ 1 & i \notin J \end{cases}$$

$\rho$  comes from the monoid map  $T \oplus \mathbb{N}^I \rightarrow T \oplus \mathbb{N}^J$  induced by the projection.  $\rho$  is an open immersion, whose image is the open substack of  $\mathcal{F}_{\underline{\mathcal{E}}}$  of objects  $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$  such that  $z_i$  generates  $\mathcal{M}_i$  for any  $i \notin J$ . We will often consider  $\mathcal{F}_{\underline{\mathcal{E}}}$  as an open substack of  $\mathcal{F}_{\underline{\mathcal{E}}}$ .

**Definition 2.18.** Given a sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s$  of elements of  $T_+^{\vee}$  we define

$$T_+^{\underline{\mathcal{E}}} = T_+^{\mathcal{E}^1, \dots, \mathcal{E}^s} = \{v \in T \mid \forall i \mathcal{E}^i(v) \geq 0\}$$

We also consider the case  $s = 0$ , so that  $T_+^{\underline{\mathcal{E}}} = T$ . If we denote by  $\hat{\phi}: T_+^{\underline{\mathcal{E}}} \rightarrow \mathbb{Z}^r$  the extension of  $\phi$ , we also define  $\mathcal{X}_{\phi}^{\underline{\mathcal{E}}} = \mathcal{Z}_{\phi}^{\underline{\mathcal{E}}} = \mathcal{X}_{\hat{\phi}}$ .

*Remark 2.19.* Assume to have a monoid map  $T_+ \rightarrow T'_+$  inducing an isomorphism on the associated groups. If  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T'^{\vee}_+ \subseteq T_+^{\vee}$ , then we have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}'_{\underline{\mathcal{E}}} & \xrightarrow{\simeq} & \mathcal{F}_{\underline{\mathcal{E}}} \\ \pi'_{\underline{\mathcal{E}}} \downarrow & & \downarrow \pi_{\underline{\mathcal{E}}} \\ \mathcal{X}_{\phi'} & \longrightarrow & \mathcal{X}_{\phi} \end{array}$$

where  $\mathcal{F}'_{\underline{\mathcal{E}}}$  is the stack obtained from  $T'_+$  with respect to  $\underline{\mathcal{E}}$ .

**Proposition 2.20.** *The map  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{X}_{\phi}$  has a natural factorization*

$$\mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{X}_{\phi}^{\underline{\mathcal{E}}} \rightarrow \mathcal{Z}_{\phi} \rightarrow \mathcal{X}_{\phi}$$

*Proof.* The factorization follows from 2.19 taking monoid maps  $T_+ \rightarrow T_+^{int} \rightarrow T_+^{\underline{\mathcal{E}}}$ .  $\square$

*Remark 2.21.* This shows that  $\pi_{\underline{\mathcal{E}}}$  has image in  $\mathcal{Z}_{\phi}$ . We will call with the same symbol  $\pi_{\underline{\mathcal{E}}}$  the factorization  $\mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{Z}_{\phi}$ .

We want now to show how the rays of  $T_+$  can be used to describe the objects of  $\mathcal{Z}_{\phi}$  over a field. Here is the result.

**Theorem 2.22.** *Let  $k$  be a field and  $T_+ \xrightarrow{a} k \in \mathcal{X}_\phi(k)$ . Then  $a \in \mathcal{Z}_\phi(k)$  if and only if there exists  $\lambda: T \rightarrow \bar{k}^*$  and  $\mathcal{E} \in T_+^\vee$  such that*

$$a(t) = \lambda_t 0^{\mathcal{E}(t)}$$

*In particular if  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$  generate  $T_+^\vee \otimes \mathbb{Q}$  then  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}}(\bar{k}) \rightarrow \mathcal{Z}_\phi(\bar{k})$  is essentially surjective and so  $\pi_{\underline{\mathcal{E}}}: |\mathcal{F}_{\underline{\mathcal{E}}}| \rightarrow |\mathcal{Z}_\phi|$  is surjective. Finally, if the map  $\phi: T \rightarrow \mathbb{Z}^r$  is injective, we have a one to one correspondence*

$$\begin{array}{ccc} \mathcal{Z}_\phi(\bar{k}) / \simeq & \xrightarrow{\gamma} & \{\text{Supp } \mathcal{E} \text{ for } \mathcal{E} \in T_+^\vee\} \\ a & \longmapsto & \{a = 0\} \end{array}$$

*In particular  $|\mathcal{Z}_\phi| = \mathcal{Z}_\phi(\overline{\mathbb{Q}}) / \simeq \bigsqcup (\bigsqcup_{\text{primes } p} \mathcal{Z}_\phi(\overline{\mathbb{F}}_p) / \simeq)$ .*

Before proving this Theorem we need some preliminary results, that we will be useful also later.

**Definition 2.23.** If  $T_+$  is integral,  $\mathcal{E} \in T_+^\vee$  and  $k$  is a field we define  $x$

$$p_{\mathcal{E}} = \bigoplus_{v \in T_+, \mathcal{E}(v) > 0} kx_v \subseteq k[T_+]$$

If  $p \in \text{Spec } k[T_+]$  we set  $p^{om} = \bigoplus_{x_v \in p} kx_v$ .

**Lemma 2.24.** *Let  $k$  be a field and assume that  $T_+$  is integral. Then:*

- (1) *if  $\mathcal{E} \in T_+^\vee$ ,  $p_{\mathcal{E}}$  is prime and  $k[\{v \in T_+ \mid \mathcal{E}(v) = 0\}] \rightarrow k[T_+] \rightarrow k[T_+]/p_{\mathcal{E}}$  is an isomorphism.*
- (2) *If  $p \in \text{Spec } k[T_+]$  then  $p^{om} = p_{\mathcal{E}}$  for some  $\mathcal{E} \in T_+^\vee$ .*

*Proof.* (1) It's obvious.

(2)  $p^{om}$  is a prime thanks to [KR05, Proposition 1.7.12] and therefore  $p^{om} = p_{\mathcal{E}}$  for some  $\mathcal{E} \in T_+^\vee$  thanks to [Ogu06, Corollary 2.2.4].  $\square$

**Remark 2.25.** If  $k$  is an algebraically closed field,  $\phi: T \rightarrow \mathbb{Z}^r$  is injective and  $a, b \in \mathcal{X}_\phi(k)$  differ by a torsor, i.e. there exists  $\lambda: T_+ \rightarrow k^*$  such that  $a = \lambda b$ , then  $a \simeq b$  in  $\mathcal{Z}_\phi(k)$ . Indeed  $\lambda$  extends to a map  $T \rightarrow k^*$  and, since  $k$  is algebraically closed, it extends again to a map  $\lambda: \mathbb{Z}^r \rightarrow k^*$ .

*Proof.* (of Theorem 2.22) We can assume that  $k$  is algebraically closed and that  $T_+$  is integral, since if  $a$  has a writing as in the statement then clearly  $a \in \mathcal{Z}_\phi(k)$ . Consider  $p = \text{Ker } a$ . Thanks to 2.24, we can write  $p^{om} = p_{\mathcal{E}}$  for some  $\mathcal{E} \in T_+^\vee$ . Set  $T'_+ = \{v \in T_+ \mid \mathcal{E}(v) = 0\}$  and  $T' = \langle T'_+ \rangle_{\mathbb{Z}}$ . Since  $a: T_+ \rightarrow k^*$ , there exists an extension  $\lambda: T' \rightarrow k^*$ . On the other hand, since  $k$  is algebraically closed, the inclusion  $T' \rightarrow T$  yields a surjection

$$\text{Hom}(T, k^*) \rightarrow \text{Hom}(T', k^*)$$

and so we can extend again to an element  $\lambda: T \rightarrow k^*$ . Since one has  $\text{Supp } \mathcal{E} = \{a = 0\}$  by construction, it is easy to check that  $a(t) = \lambda_t 0^{\mathcal{E}(t)}$  for any  $t \in T_+$ .

Now consider the last part of the statement and so assume  $\phi: T \rightarrow \mathbb{Z}^r$  injective. The map  $\gamma$  is well defined thanks to above and surjective since, given  $\mathcal{E} \in T_+^\vee$ , one can always define  $a(t) = 0^{\mathcal{E}(t)}$ . For the injectivity, let  $a, b \in \mathcal{Z}_\phi(k)$  such that  $\{a = 0\} = \{b = 0\}$ . We can write  $a(t) = \lambda_t 0^{\mathcal{E}(t)}$ ,  $b(t) = \mu_t 0^{\mathcal{E}(t)}$ , where  $\lambda, \mu: T \rightarrow k^*$ , so that  $a, b$  differ by a torsor and are therefore isomorphic thanks to 2.25. Finally, since any point of  $|\mathcal{Z}_\phi|$  comes from  $\mathbb{Z}$ , we also have the last equality.  $\square$

In some cases the description of the objects of  $\mathcal{F}_{\underline{\mathcal{E}}}$  can be simplified, regardless of  $\underline{\mathcal{E}}$ , in the sense that there exist a stack of reduced data  $\mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}}$ , whose objects can be described by less data, and an isomorphism  $\mathcal{F}_{\underline{\mathcal{E}}} \simeq \mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}}$ . This kind of

simplification could be very useful when we have to deal with an explicit map of monoid  $\phi: T_+ \rightarrow \mathbb{Z}^r$ , as we will see in 3.7. The idea is that in order to define an object  $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}$  we don't really need all the invertible sheaves  $\mathcal{L}_1, \dots, \mathcal{L}_r$ , because they are uniquely determined by a subset of them and the other data.

**Definition 2.26.** Assume  $T \xrightarrow{\phi} \mathbb{Z}^r$  injective. Let  $V \subseteq \mathbb{Z}^r$  be a submodule with a given basis  $v_1, \dots, v_q$  and  $\sigma: \mathbb{Z}^r \rightarrow V$  be a map such that  $(\text{id} - \sigma)\mathbb{Z}^r \subseteq T$  (or equivalently  $\pi = \pi \circ \sigma$  where  $\pi$  is the projection  $\mathbb{Z}^r \rightarrow \text{Coker } \phi$ ). Define  $W = \langle (\text{id} - \sigma)V, \sigma T \rangle$ . Given  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^l \in T_+^\vee$  consider the map

$$\begin{aligned} W \oplus \mathbb{N}^s &\xrightarrow{\psi_{\underline{\mathcal{E}}, \sigma}} \mathbb{Z}^q \oplus \mathbb{Z}^s \\ (w, z) &\longmapsto (-w, \underline{\mathcal{E}}(w) + z) \end{aligned}$$

We define  $\mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}, \sigma} = \mathcal{X}_{\psi_{\underline{\mathcal{E}}, \sigma}}$  and we call it the stack of reduced data of  $\underline{\mathcal{E}}$ .

**Lemma 2.27.** Consider a submodule  $U \subseteq \mathbb{Z}^p$ , a map  $\underline{\mathcal{E}}: U \rightarrow \mathbb{Z}^l$  and  $\tau: \mathbb{Z}^p \rightarrow \mathbb{Z}^p$  such that  $(\text{id} - \tau)\mathbb{Z}^p \subseteq U$ . Consider the commutative diagram

$$\begin{array}{ccccc} (u, z) & & U \oplus \mathbb{N}^l & \xrightarrow{\tau \oplus \text{id}} & U \oplus \mathbb{N}^l \\ \downarrow & & \downarrow \psi & & \downarrow \psi \\ (-u, \underline{\mathcal{E}}(u) + z) & & \mathbb{Z}^p \oplus \mathbb{Z}^l & \longrightarrow & \mathbb{Z}^p \oplus \mathbb{Z}^l \\ & & (u, z) \longmapsto & & (\tau u, \underline{\mathcal{E}}(u - \tau u) + z) \end{array}$$

Then the induced map  $\varphi: \mathcal{X}_{\psi} \rightarrow \mathcal{X}_{\psi}$  is isomorphic to  $\text{id}_{\mathcal{X}_{\psi}}$ .

*Proof.* Let  $x_1, \dots, x_p$  be a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^p$  with  $a_1, \dots, a_k \in \mathbb{N}$  such that  $a_1 x_1, \dots, a_k x_k$  is a  $\mathbb{Z}$ -basis of  $U$ . We want to define a natural isomorphism  $\text{id}_{\mathcal{X}_{\psi}} \xrightarrow{\omega} \varphi$ . First note that it is enough to define it on the objects of  $\mathcal{X}_{\psi}$  coming from the atlas  $\text{Spec } \mathbb{Z}[U \oplus \mathbb{N}^l]$ , prove the naturality between such objects on a fixed scheme  $T$  and for the restrictions. An object coming from the atlas is of the form  $(\lambda, \underline{z})$  where  $\lambda: U \rightarrow \mathcal{O}_T^*$  is an additive map and  $\underline{z} = z_1, \dots, z_l \in \mathcal{O}_T$ . Moreover  $\varphi(\lambda, \underline{z}) = (\tilde{\lambda}, \underline{z})$  where  $\tilde{\lambda} = \lambda \circ \tau$ . Let  $\underline{\eta} \in D(\mathbb{Z}^p)(T)$  the only elements such that  $\underline{\eta}^{x_i} = \lambda(x_i - \tau x_i)$  for  $i = 1, \dots, p$ . These objects are well defined since  $(\text{id} - \tau)\mathbb{Z}^p \subseteq U$ . We claim that  $\omega_{T, (\lambda, \underline{z})} = (\underline{\eta}, \underline{1})$  is an isomorphism  $(\lambda, \underline{z}) \rightarrow \varphi(\lambda, \underline{z})$  and define a natural transformation. It is an isomorphism since  $1z_j = z_j$  and the condition

$$\underline{\eta}^{-u} \underline{1}^{\underline{\mathcal{E}}(u)} \lambda(u) = \lambda(\tau u) \quad \forall u \in U$$

holds by construction checking it on the basis  $a_1 x_1, \dots, a_k x_k$  of  $U$  (see 2.5). It's also easy to check that this isomorphisms commute with the change of basis. So it remains to prove that, if  $(\underline{\sigma}, \underline{\mu})$  is an isomorphism  $(\lambda, \underline{z}) \rightarrow (\lambda', \underline{z}')$  then we have a commutative diagram

$$\begin{array}{ccc} (\lambda, \underline{z}) & \xrightarrow{(\underline{\sigma}, \underline{\mu})} & (\lambda', \underline{z}') \\ \omega_{T, (\lambda, \underline{z})} \downarrow & & \downarrow \omega_{T, (\lambda', \underline{z}')} \\ \varphi(\lambda, \underline{z}) & \xrightarrow{\varphi(\underline{\sigma}, \underline{\mu})} & \varphi(\lambda', \underline{z}') \end{array}$$

We have  $\varphi(\underline{\sigma}, \underline{\mu}) = (\tilde{\underline{\sigma}}, \tilde{\underline{\mu}})$  with  $\tilde{\underline{\mu}} = \underline{\mu}$  and  $\tilde{\underline{\sigma}}^{x_i} = \underline{\sigma}^{\tau x_i} \underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)}$  (see 2.7). So it is easy to check that the commutativity in the second member holds. For the first, the condition is  $\tilde{\underline{\sigma}} \underline{\eta} = \underline{\eta}' \underline{\sigma}$ , which is equivalent to

$$(\tilde{\underline{\sigma}} \underline{\eta})^{x_i} = \underline{\sigma}^{\tau x_i} \underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)} \lambda(x_i - \tau x_i) = (\underline{\eta}' \underline{\sigma})^{x_i} = \lambda'(x_i - \tau x_i) \underline{\sigma}^{x_i}$$

and to  $\underline{\sigma}^{-(x_i - \tau x_i)} \underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)} \lambda(x_i - \tau x_i) = \lambda'(x_i - \tau x_i)$  for any  $i$ . But, since  $(\underline{\sigma}, \underline{\mu})$  is an isomorphism  $(\lambda, \underline{z}) \rightarrow (\lambda', \underline{z}')$ , the condition

$$\underline{\sigma}^{-u} \underline{\mu}^{\underline{\mathcal{E}}(u)} \lambda(u) = \lambda'(u) \quad \forall u \in U$$

has to be satisfied.  $\square$

**Proposition 2.28.** *Assume  $T \xrightarrow{\phi} \mathbb{Z}^r$  injective and let  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^\vee$  and  $\sigma, V, v_1, \dots, v_q$  be as in 2.26. Then we have functors*

$$\begin{array}{ccc} ((\underline{\mathcal{N}}^{\sigma e_i} \otimes \underline{\mathcal{M}}^{\underline{\mathcal{E}}(e_i - \sigma e_i)})_{i=1, \dots, r}, \underline{\mathcal{M}}, \underline{z}, \tilde{\lambda}) & \longleftarrow & (\underline{\mathcal{N}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \\ \mathcal{F}_{\underline{\mathcal{E}}} & \xrightarrow{\quad} & \mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}, \sigma} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) & \longmapsto & ((\underline{\mathcal{L}}^{v_i})_{i=1, \dots, q}, \underline{\mathcal{M}}, \underline{z}, \lambda|_W) \end{array}$$

for appropriate choices of  $\tilde{\lambda}$  that are inverses of each other.

*Proof.* Consider the commutative diagrams

$$\begin{array}{ccc} W \oplus \mathbb{N}^s & \hookrightarrow & T \oplus \mathbb{N}^s \\ \psi \downarrow & & \downarrow \phi_{\underline{\mathcal{E}}} \\ \mathbb{Z}^q \oplus \mathbb{Z}^s & \hookrightarrow & \mathbb{Z}^r \oplus \mathbb{Z}^s \end{array} \quad \begin{array}{ccc} T \oplus \mathbb{N}^s & \xrightarrow{\sigma \oplus \text{id}} & W \oplus \mathbb{N}^s \\ \phi_{\underline{\mathcal{E}}} \downarrow & & \downarrow \psi \\ \mathbb{Z}^r \oplus \mathbb{Z}^s & \xrightarrow{\quad} & \mathbb{Z}^q \oplus \mathbb{Z}^s \\ (x, y) & \longmapsto & (\sigma x, \underline{\mathcal{E}}(x - \sigma x) + y) \end{array}$$

They induce functors  $\Lambda: \mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}, \sigma}$  and  $\Delta: \mathcal{F}_{\underline{\mathcal{E}}}^{\text{red}, \sigma} \rightarrow \mathcal{F}_{\underline{\mathcal{E}}}$  respectively, that behave as the functors of the statement thanks to description given in 2.6. Finally, applying 2.27, we obtain that  $\Lambda \circ \Delta \simeq \text{id}$  and  $\Delta \circ \Lambda \simeq \text{id}$ .  $\square$

**2.2. Integral extremal rays and smooth sequences.** We continue to use notation from 2.8. We have seen that given a collection  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^\vee$  we can associate to it a stack  $\mathcal{F}_{\underline{\mathcal{E}}}$  and a 'parametrization' map  $\mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{X}_\phi$ . The stack  $\mathcal{F}_{\underline{\mathcal{E}}}$  could be 'too big' if we don't make an appropriate choice of the collection  $\underline{\mathcal{E}}$ . This happens for example if the rays in  $\underline{\mathcal{E}}$  are not distinct or, more generally, if a ray in  $\underline{\mathcal{E}}$  belongs to the submonoid generated by the other rays in  $\underline{\mathcal{E}}$ . Thus we want to restrict our attention to a special class of integral rays, called extremal and to special sequences of them.

**Definition 2.29.** An integral *extremal* ray for  $T_+$  is an element  $\mathcal{E} \in T_+^\vee$  such that

- $\mathcal{E}$  is minimal for the condition  $\text{Supp } \mathcal{E} \neq \emptyset$ ;
- $\mathcal{E}$  is normalized, i.e.  $\mathcal{E}: T \rightarrow \mathbb{Z}$  is surjective.

**Lemma 2.30.** *Assume that  $T_+$  is an integral monoid and let  $v_1, \dots, v_l$  be a system of generators of  $T_+$ . Then the integral extremal rays are the normalized  $\mathcal{E} \in T_+^\vee - \{0\}$  such that  $\text{Ker } \mathcal{E}$  contains  $\text{rk } T - 1$   $\mathbb{Q}$ -independent vectors among the  $v_1, \dots, v_l$ . In particular they are finitely many and they generate  $\mathbb{Q}_+ T_+^\vee$ .*

*Proof.* Denote by  $\Omega \subseteq T_+^\vee$  the set of elements defined in the statement. From [Ful93, Section 1.2, (9)] it follows that  $\mathbb{Q}_+ \Omega = \mathbb{Q}_+ T_+^\vee$ . If  $\mathcal{E} \in \Omega$  then it is an integral extremal ray. Indeed

$$\emptyset \neq \text{Supp } \mathcal{E}' \subseteq \text{Supp } \mathcal{E} \implies \exists \lambda \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E}' = \lambda \mathcal{E} \implies \text{Supp } \mathcal{E}' = \text{Supp } \mathcal{E}$$

Conversely let  $\mathcal{E}$  be an integral extremal ray and consider a writing

$$\mathcal{E} = \sum_{\delta \in \Omega} \lambda_\delta \delta \quad \text{with } \lambda_\delta \in \mathbb{Q}_{\geq 0}$$

There must exists  $\delta$  such that  $\lambda_\delta \neq 0$ . So

$$\text{Supp } \delta \subseteq \text{Supp } \mathcal{E} \implies \text{Supp } \delta = \text{Supp } \mathcal{E} \implies \exists \mu \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E} = \mu \delta \implies \mathcal{E} = \delta$$

□

**Corollary 2.31.** *For an integral extremal ray  $\mathcal{E}$  and  $\mathcal{E}' \in T_+^\vee$  we have*

$$\text{Supp } \mathcal{E}' = \text{Supp } \mathcal{E} \iff \exists \lambda \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E}' = \lambda \mathcal{E} \iff \exists \lambda \in \mathbb{N}_+ \text{ s.t. } \mathcal{E}' = \lambda \mathcal{E}$$

**Definition 2.32.** An element  $v \in T_+$  is said *indecomposable* if whenever  $v = v' + v''$  with  $v', v'' \in T_+$  it follows that  $v' = 0$  or  $v'' = 0$ .

**Proposition 2.33.**  $T_+^\vee$  has a unique minimal system of generators composed by the indecomposable elements. Moreover any integral extremal ray is indecomposable.

*Proof.* The first claim of the statement follows from [Ogu06, Proposition 2.1.2] since  $T_+^\vee$  is sharp, i.e. doesn't contain invertible elements. For the second consider an extremal ray  $\mathcal{E}$  and assume  $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$ . We have

$$\text{Supp } \mathcal{E}', \text{Supp } \mathcal{E}'' \subseteq \text{Supp } \mathcal{E} \implies \mathcal{E}' = \lambda \mathcal{E}, \mathcal{E}'' = \mu \mathcal{E} \text{ with } \lambda, \mu \in \mathbb{N}$$

and so  $\mathcal{E} = (\lambda + \mu)\mathcal{E} \implies \lambda + \mu = 1 \implies \lambda = 0$  or  $\mu = 0 \implies \mathcal{E}' = 0$  or  $\mathcal{E}'' = 0$ . □

**Definition 2.34.** A *smooth sequence* for  $T_+$  is a sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$  for which there exist elements  $v_1, \dots, v_s$  in the associated integral monoid  $T_+^{int}$  of  $T_+$  such that

$$T_+^{int} \cap \text{Ker } \underline{\mathcal{E}} \text{ generates } \text{Ker } \underline{\mathcal{E}} \quad \text{and} \quad \mathcal{E}^i(v_j) = \delta_{i,j}$$

We will also say that a ray  $\mathcal{E} \in T_+^\vee - \{0\}$  is *smooth* if there exists a smooth sequence as above such that  $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^s \rangle_{\mathbb{N}}$  or, equivalently, such that  $\text{Supp } \mathcal{E} \subseteq \text{Supp } \underline{\mathcal{E}}$ .

*Remark 2.35.* If  $T_+$  is integral and  $\Omega$  is a system of generators of it, one can always assume that  $v_i \in \Omega$ . Moreover we also have that  $\Omega \cap \text{Ker } \underline{\mathcal{E}}$  generates  $\text{Ker } \underline{\mathcal{E}}$ .

Finally the “equivalently” in definition 2.34 follows from the fact that, since  $\text{Ker } \underline{\mathcal{E}}$  is generated by elements in  $T_+^{int}$ , then the inclusion of the supports implies that  $\mathcal{E}|_{\text{Ker } \underline{\mathcal{E}}} = 0$  and therefore  $\mathcal{E} = \sum_i \mathcal{E}(v_i) \mathcal{E}^i$ .

**Lemma 2.36.** *Let  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$  be a smooth sequence. Then*

$$T_+^{\underline{\mathcal{E}}} = \text{Ker } \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r \rangle_{\mathbb{N}} \subseteq T \text{ where } v_1, \dots, v_r \in T_+^{int}, \mathcal{E}^i(v_j) = \delta_{i,j}$$

Moreover, if  $z_1, \dots, z_s \in T_+^{int}$  generate  $T_+^{int}$ , then  $\mathbb{Z}[T_+^{\underline{\mathcal{E}}}] = \mathbb{Z}[T_+^{int}]_{\prod_{\mathcal{E}(z_i)=0} x_{z_i}}$  so that  $\text{Spec } \mathbb{Z}[T_+^{\underline{\mathcal{E}}}] (\mathcal{X}_\phi^{\underline{\mathcal{E}}})$  is a smooth open subscheme (substack) of  $\text{Spec } \mathbb{Z}[T_+^{int}] (\mathcal{Z}_\phi)$ .

*Proof.* We have  $T = \text{Ker } \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r \rangle_{\mathbb{Z}}$  and clearly  $\text{Ker } \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q \rangle_{\mathbb{N}} \subseteq T_+^{\underline{\mathcal{E}}}$ . Conversely if  $v \in T_+^{\underline{\mathcal{E}}}$  we can write

$$v = z + \sum_i \mathcal{E}^i(v) v_i \text{ with } z \in \text{Ker } \underline{\mathcal{E}} \implies v \in \text{Ker } \underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q \rangle_{\mathbb{N}}$$

In particular  $\text{Spec } \mathbb{Z}[T_+^{\underline{\mathcal{E}}}] \simeq \mathbb{A}_{\mathbb{Z}}^r \times D_{\mathbb{Z}}(\text{Ker } \underline{\mathcal{E}})$  and so both  $\text{Spec } \mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$  and  $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$  are smooth. Now let

$$I = \{i \mid \mathcal{E}(z_i) = 0\} \text{ and } S_+ = \langle T_+^{int}, -z_i \text{ for } i \in I \rangle_{\mathbb{Z}} \subseteq T$$

We need to prove that  $S_+ = T_+^{\underline{\mathcal{E}}}$ . Clearly we have the inclusion  $\subseteq$ . For the other one, it is enough to prove that  $-\text{Ker } \underline{\mathcal{E}} \cap T_+^{int} \subseteq S_+$ . But if  $v \in \text{Ker } \underline{\mathcal{E}} \cap T_+^{int}$  then

$$v = \sum_{j=1}^s a_j z_j = \sum_{j \in I} a_j z_j \implies -v \in S_+$$

□

*Remark 2.37.* Any subsequence of a smooth sequence is smooth too. Indeed let  $\underline{\delta} = \mathcal{E}^1, \dots, \mathcal{E}^s$  a subsequence of a smooth sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ , with  $r > s$ . We have to prove that  $\langle \text{Ker } \underline{\delta} \cap T_+^{\text{int}} \rangle_{\mathbb{Z}} = \text{Ker } \underline{\delta}$ . Take  $v \in \text{Ker } \underline{\delta}$ . So

$$v - \sum_{j=s+1}^r \mathcal{E}^j(v) v_j \in \text{Ker } \underline{\mathcal{E}} = \langle \text{Ker } \underline{\mathcal{E}} \cap T_+^{\text{int}} \rangle_{\mathbb{Z}} \subseteq \langle \text{Ker } \underline{\delta} \cap T_+^{\text{int}} \rangle_{\mathbb{Z}} \implies v \in \langle \text{Ker } \underline{\delta} \cap T_+^{\text{int}} \rangle_{\mathbb{Z}}$$

**Proposition 2.38.** *Let  $\mathcal{E} \in T_+^{\vee}$ . Then  $\mathcal{E}$  is a smooth integral extremal ray if and only if  $\mathcal{E}$  is a smooth sequence of one element, i.e.  $\text{Ker } \mathcal{E} \cap T_+^{\text{int}}$  generates  $\text{Ker } \mathcal{E}$  and there exists  $v \in T_+$  such that  $\mathcal{E}(v) = 1$ .*

*In particular any element of a smooth sequence is a smooth integral extremal ray.*

*Proof.* We can assume  $T_+$  integral. If  $\mathcal{E}$  is smooth and extremal, then there exists a smooth sequence  $\mathcal{E}^1, \dots, \mathcal{E}^q$  such that  $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}}$ . Since  $\mathcal{E}$  is indecomposable, it follows that  $\mathcal{E} = \mathcal{E}^i$  for some  $i$ . Conversely assume that  $\mathcal{E}$  is a smooth sequence. So it is smooth by definition and it is normalized since  $\mathcal{E}(v) = 1$  for some  $v$ . Finally an inclusion  $\text{Supp } \delta \subseteq \text{Supp } \mathcal{E}$  for  $\delta \in T_+^{\vee}$  means that  $\delta \in \langle \mathcal{E} \rangle_{\mathbb{N}}$ , as remarked in 2.35, and so  $\text{Supp } \delta = \emptyset$  or  $\text{Supp } \delta = \text{Supp } \mathcal{E}$ .  $\square$

We conclude with a lemma that will be useful later.

**Lemma 2.39.** *Let  $T_+, T'_+$  be integral monoids and  $h: T \rightarrow T'$  be an homomorphism such that  $h(T_+) = T'_+$  and  $\text{Ker } h = \langle \text{Ker } h \cap T_+ \rangle$ . If  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^{\vee}$  then*

$$\underline{\mathcal{E}} \text{ smooth sequence for } T'_+ \iff \underline{\mathcal{E}} \circ h \text{ smooth sequence for } T_+$$

*Proof.* Clearly there exist  $v_i \in T'_+$  such that  $\mathcal{E}^i(v_j) = \delta_{i,j}$  if and only if there exist  $w_i \in T_+$  such that  $\mathcal{E}^i \circ h(w_j) = \delta_{i,j}$ . On the other hand we have a surjective morphism

$$\text{Ker } \underline{\mathcal{E}} \circ h / \langle \text{Ker } \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}} \longrightarrow \text{Ker } \underline{\mathcal{E}} / \langle \text{Ker } \underline{\mathcal{E}} \cap T'_+ \rangle_{\mathbb{Z}}$$

In order to conclude it is enough to prove that this map is injective. So let  $v \in T$  such that

$$h(v) = \sum_j a_j z_j \text{ with } a_j \in \mathbb{Z}, z_j \in T'_+, \underline{\mathcal{E}}(z_j) = 0$$

Since  $h(T_+) = T'_+$ , there exist  $y_j \in T_+$  such that  $h(y_j) = z_j$ . In particular  $y = \sum_j a_j y_j \in \langle \text{Ker } \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}}$  and

$$v - y \in \text{Ker } h = \langle \text{Ker } h \cap T_+ \rangle \subseteq \langle \text{Ker } \underline{\mathcal{E}} \circ h \cap T_+ \rangle$$

$\square$

### 2.3. The smooth locus $\mathcal{Z}_{\phi}^{\text{sm}}$ of the main component $\mathcal{Z}_{\phi}$ .

**Lemma 2.40.** *Let  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^q$  be a smooth sequence and  $\chi$  be a finite sequence of elements of  $T_+^{\vee}$ . Assume that all the elements of  $\chi$  are distinct, each  $\mathcal{E}^i$  is an element of  $\chi$  and that for any  $\delta$  in  $\chi$  we have*

$$\delta \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \implies \exists i \delta = \mathcal{E}^i$$

*As usual denote by  $\pi_{\chi}$  the map  $\mathcal{F}_{\chi} \rightarrow \mathcal{X}_{\phi}$ . Then we have an equivalence*

$$\mathcal{F}_{\underline{\mathcal{E}}} = \pi_{\chi}^{-1}(\mathcal{X}_{\phi}^{\underline{\mathcal{E}}}) \xrightarrow{\sim} \mathcal{X}_{\phi}^{\underline{\mathcal{E}}}$$

*Proof.* Set  $\chi = \mathcal{E}^1, \dots, \mathcal{E}^q, \eta^1, \dots, \eta^l = \underline{\mathcal{E}}, \underline{\eta}$ . We first prove that  $\pi_{\chi}^{-1}(\mathcal{X}_{\phi}^{\underline{\mathcal{E}}}) \subseteq \mathcal{F}_{\underline{\mathcal{E}}}$ . Since they are open substacks, we can check this over an algebraically closed field  $k$ . Let  $(z, \lambda) \in \pi_{\chi}^{-1}(\mathcal{X}_{\phi}^{\underline{\mathcal{E}}})$  so that  $a = \pi_{\chi}(z, \lambda) = z^{\underline{\mathcal{E}}}/\lambda: T_+ \rightarrow k$  by 2.16. We have to prove that  $z_{\eta_j} \neq 0$ . Assume by contradiction that  $z_{\eta_j} = 0$ . Since we can write



$a = b0^{\eta_j}$  and since  $a$  extends to  $T_+^{\underline{\mathcal{E}}}$  so that  $a(t) \neq 0$  if  $t \in T_+ \cap \text{Ker } \underline{\mathcal{E}}$ , we have that  $\eta_j$  is 0 on  $T_+ \cap \text{Ker } \underline{\mathcal{E}}$ . In particular

$$\text{Supp } \eta^j \subseteq \text{Supp } \underline{\mathcal{E}} \implies \eta^j \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \implies \exists i \eta^j = \mathcal{E}^i$$

Thanks to 2.17, we can reduce to prove that if  $\underline{\mathcal{E}}$  is a smooth sequence such that  $T_+ = T_+^{\underline{\mathcal{E}}}$  then  $\pi_{\underline{\mathcal{E}}}$  is an isomorphism. By 2.36 we can write  $T_+ = W \oplus \mathbb{N}^q$ , where  $W$  is a free  $\mathbb{Z}$ -module such that  $\underline{\mathcal{E}}|_W = 0$  and, if we denote by  $v_1, \dots, v_q$  the canonical base of  $\mathbb{N}^q$ ,  $\mathcal{E}^j(v_i) = \delta_{i,j}$ . Consider the diagram

$$\begin{array}{ccc} \mathbb{N}^q \oplus T & & T_+ \\ \parallel & & \parallel \\ \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q & \xrightarrow{\gamma} & W \oplus \mathbb{N}^q \\ \sigma_{\underline{\mathcal{E}}} \downarrow & & \downarrow \phi \\ \mathbb{Z}^q \oplus \mathbb{Z}^r & \xrightarrow{\delta} & \mathbb{Z}^r \end{array} \quad \begin{array}{l} \gamma(e_i) = v_i, \gamma|_W = -\text{id}_W, \gamma(v_i) = 0 \\ \delta(e_i) = \phi(v_i), \delta|_{\mathbb{Z}^r} = \text{id}_{\mathbb{Z}^r} \end{array}$$

One can check directly its commutativity. In this way we get a map  $s: \mathcal{X}_{\phi} \rightarrow \mathcal{F}_{\underline{\mathcal{E}}}$ . Again a direct computation on the diagrams defining  $s$  and  $\pi_{\underline{\mathcal{E}}}$  shows that  $\pi_{\underline{\mathcal{E}}} \circ s \simeq \text{id}_{\mathcal{X}_{\phi}}$  and that the diagram inducing  $G = s \circ \pi_{\underline{\mathcal{E}}}$  is

$$\begin{array}{ccc} \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q & \xrightarrow{\alpha} & \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \\ \sigma_{\underline{\mathcal{E}}} \downarrow & & \downarrow \sigma_{\underline{\mathcal{E}}} \\ \mathbb{Z}^q \oplus \mathbb{Z}^r & \xrightarrow{\beta} & \mathbb{Z}^q \oplus \mathbb{Z}^r \end{array} \quad \begin{array}{l} \alpha(e_i) = e_i - v_i, \alpha|_W = \text{id}_W, \alpha|_{\mathbb{Z}^q} = 0 \\ \beta(e_i) = \phi(v_i), \beta|_{\mathbb{Z}^r} = \text{id}_{\mathbb{Z}^r} \end{array}$$

We will prove that  $G \simeq \text{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$ . An object of  $\mathcal{F}_{\underline{\mathcal{E}}}(A)$ , where  $A$  is a ring, coming from the atlas is given by  $a = (\underline{z}, \lambda, \underline{\mu}): \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \rightarrow A$  where  $\underline{z} = (a(e_i))_i = z_1, \dots, z_q \in A$ ,  $\lambda = a|_W: W \rightarrow A^*$  is an homomorphism and  $\underline{\mu} = (\mu(v_i))_i = \mu_1, \dots, \mu_q \in A^*$ . Moreover  $Ga = a \circ \alpha = ((z_i/\mu_i)_i, \lambda, \underline{1})$ . It is now easy to check that  $(\underline{\mu}, 1): Ga \rightarrow a$  is an isomorphism and that this map defines an isomorphism  $G \rightarrow \text{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$ .  $\square$

**Corollary 2.41.** *If  $\underline{\mathcal{E}}$  is a smooth sequence then  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \rightarrow \mathcal{Z}_{\phi}$  is an open immersion with image  $\mathcal{X}_{\phi}^{\underline{\mathcal{E}}}$ .*

It turns out that if  $\underline{\mathcal{E}}$  is a smooth sequence, then  $\mathcal{X}_{\phi}^{\underline{\mathcal{E}}}$  has a more explicit description:

**Proposition 2.42.** *Let  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$  be a smooth sequence,  $k$  be a field and  $a \in \mathcal{X}_{\phi}(k)$*

$$a \in \mathcal{X}_{\phi}^{\underline{\mathcal{E}}}(k) \iff \exists \mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}, \lambda: T \rightarrow \overline{k}^* \text{ s.t. } a = \lambda 0^{\mathcal{E}}$$

*Moreover if  $\lambda 0^{\mathcal{E}} \in \mathcal{X}_{\phi}^{\underline{\mathcal{E}}}(k)$ , for some  $\mathcal{E} \in T_+^{\vee}$ ,  $\lambda: T \rightarrow \overline{k}^*$ , then  $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$ .*

*Proof.* We can assume  $k$  algebraically closed and  $T_+$  integral. In this case  $a \in \mathcal{X}_{\phi}^{\underline{\mathcal{E}}}(k)$  if and only if  $a: T_+ \rightarrow k$  extends to a map  $\text{Ker } \underline{\mathcal{E}} \oplus \mathbb{N}^r = T_+^{\underline{\mathcal{E}}} \rightarrow k$ . So  $\Leftarrow$  holds. Conversely, from 2.22, we can write  $a = \lambda 0^{\mathcal{E}}$  where  $\lambda: T \rightarrow k^*$  and  $\mathcal{E} \in (T_+^{\underline{\mathcal{E}}})^{\vee}$ . From 2.36 we see that  $T_+^{\underline{\mathcal{E}}} = \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$ . Finally, if  $\lambda 0^{\mathcal{E}} \in \mathcal{X}_{\phi}^{\underline{\mathcal{E}}}$  for some  $\mathcal{E}$ , then  $\text{Supp } \mathcal{E} \subseteq \text{Supp } \underline{\mathcal{E}}$  and we have done.  $\square$

**Lemma 2.43.** *Let  $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$  be a sequence of distinct smooth extremal rays and  $\Theta$  be a collection of smooth sequences with rays in  $\underline{\mathcal{E}}$ . Set*

$$\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}} \mid \begin{array}{l} V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \neq \emptyset \\ \text{iff } \exists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\}$$

*Then, taking into account the identification made in 2.17, we have*

$$\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\delta}}$$

*Proof.* Let  $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\delta}}(T)$ , for some scheme  $T$  and let  $p \in V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$ . This means that the pullback of  $\pi_{\underline{\mathcal{E}}}(\chi)$  to  $\overline{k(p)}$  is given by  $a = b0^{\mathcal{E}^{i_1} + \cdots + \mathcal{E}^{i_s}}$  for some  $b: T_+ \rightarrow \overline{k(p)}$ . By definition there exists  $\underline{\delta} \in \Theta$  such that  $a \in \mathcal{F}_{\underline{\delta}}(\overline{k(p)})$ , i.e.  $a = \mu 0^{\delta}$  for some  $\delta \in \langle \underline{\delta} \rangle_{\mathbb{N}}$ ,  $\mu: T \rightarrow \overline{k(p)}^*$ . So

$$\text{Supp } \mathcal{E}^{i_j} \subseteq \{a = 0\} = \text{Supp } \delta \subseteq \text{Supp } \underline{\delta} \implies \mathcal{E}^{i_j} \in \langle \underline{\delta} \rangle_{\mathbb{N}}$$

For the other inclusion, since all the  $\mathcal{F}_{\underline{\delta}}$  are open substacks of  $\mathcal{F}_{\underline{\mathcal{E}}}$ , we can reduce to the case of an algebraically closed field  $k$ . So let  $(\underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}(k)$  and set  $J = \{i \in I \mid z_i = 0\}$ . By definition of  $\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}$  there exists  $\underline{\delta} \in \Theta$  such that  $\underline{\eta} = (\mathcal{E}^j)_{j \in J} \subseteq \underline{\delta}$  and, taking into account 2.17, this means that  $a \in \mathcal{F}_{\underline{\eta}}(k) \subseteq \mathcal{F}_{\underline{\delta}}(k)$ .  $\square$

**Definition 2.44.** Let  $\Theta$  be a collection of smooth sequences. We define

$$X_{\phi}^{\Theta} = \bigcup_{\underline{\delta} \in \Theta} \text{Spec } \mathbb{Z}[T_+^{\underline{\delta}}] \subseteq \text{Spec } \mathbb{Z}[T_+] \text{ and } \mathcal{X}_{\phi}^{\Theta} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{X}_{\phi}^{\underline{\delta}} \subseteq \mathcal{Z}_{\phi}$$

**Theorem 2.45.** *Let  $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$  be a sequence of distinct smooth integral rays and  $\Theta$  be a collection of smooth sequences with rays in  $\underline{\mathcal{E}}$ . Then we have an isomorphism*

$$\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{X}_{\phi}^{\Theta}) \xrightarrow{\sim} \mathcal{X}_{\phi}^{\Theta}$$

*Proof.* Taking into account 2.43, it is enough to note that

$$\pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{X}_{\phi}^{\Theta}) = \pi_{\underline{\mathcal{E}}}^{-1}\left(\bigcup_{\underline{\delta} \in \Theta} \mathcal{X}_{\phi}^{\underline{\delta}}\right) = \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\mathcal{E}} \cap \underline{\delta}} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\delta}} \xrightarrow{\sim} \mathcal{X}_{\phi}^{\Theta}$$

$\square$

**Proposition 2.46.** *Let  $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$  be a sequence of distinct smooth integral rays and  $\Theta$  be a collection of smooth sequences with rays in  $\underline{\mathcal{E}}$ . Then the set*

$$\Delta^{\Theta} = \{\langle \eta_1, \dots, \eta_r \rangle_{\mathbb{Q}_+} \mid \exists \underline{\delta} \in \Theta \text{ s.t. } \eta_1, \dots, \eta_r \subseteq \underline{\delta}\}$$

*is a toric fan in  $T^{\vee} \otimes \mathbb{Q}$  whose associated toric variety over  $\mathbb{Z}$  is  $X_{\phi}^{\Theta}$ . Moreover*

$$\mathcal{X}_{\phi}^{\Theta} \simeq [X_{\phi}^{\Theta}/D(\mathbb{Z}^r)]$$

*Proof.* We know that if  $\underline{\eta}$  is a smooth sequence then  $\text{Spec } \mathbb{Z}[T_+^{\underline{\eta}}]$  is a smooth open subset of  $\text{Spec } \mathbb{Z}[T_+^{int}]$  and it is the affine toric variety associated to the cone  $\langle \underline{\eta} \rangle_{\mathbb{Q}_+}$ . It is then easy to check that  $\Delta^{\Theta}$  is a fan whose associated toric variety is  $X_{\phi}^{\Theta}$ . Since  $\text{Spec } \mathbb{Z}[T_+^{\underline{\eta}}]$  is the equivariant open subset of  $\text{Spec } \mathbb{Z}[T_+^{int}]$  inducing  $\mathcal{X}_{\phi}^{\underline{\eta}}$  in  $\mathcal{Z}_{\phi}$ , then  $X_{\phi}^{\Theta}$  is the equivariant open subset of  $\text{Spec } \mathbb{Z}[T_+^{int}]$  inducing  $\mathcal{X}_{\phi}^{\Theta}$ . In particular we obtain the last isomorphism.  $\square$

**Lemma 2.47.** *Assume  $T_+$  integral and set  $\Theta$  for the set of all smooth sequences. Then  $X_{\phi}^{\Theta}$  is the smooth locus of  $\text{Spec } \mathbb{Z}[T_+]$ . In particular  $\mathcal{Z}_{\phi}^{sm} = \mathcal{X}_{\phi}^{\Theta} \simeq [X_{\phi}^{\Theta}/D(\mathbb{Z}^r)]$ .*

*Proof.* From 2.36 we know that  $\text{Spec } \mathbb{Z}[T_+^{\mathcal{E}}]$  is smooth over  $\mathbb{Z}$  and it is an open subset of  $\text{Spec } \mathbb{Z}[T_+]$ . So we focus on the converse. Since  $\text{Spec } \mathbb{Z}[T_+]$  is flat over  $\mathbb{Z}$ , we can replace  $\mathbb{Z}$  by an algebraically closed field  $k$ . Let  $p \in \text{Spec } k[T_+]$  be a smooth point. In particular  $p^{om}$  is smooth too. If  $p^{om} = 0$  then  $p \in \text{Spec } k[T]$  and we have done. So we can assume  $p^{om} = p_{\mathcal{E}}$  for some  $0 \neq \mathcal{E} \in T_+^{\vee}$  thanks to 2.24. We claim that there exist a smooth sequence  $\mathcal{E}^1, \dots, \mathcal{E}^q$  such that  $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}}$ . This is enough to conclude that  $p \in \text{Spec } k[T_+^{\mathcal{E}}]$ . Indeed if  $x_w \in p$  for some  $w \in \text{Ker } \underline{\mathcal{E}} \cap T_+$  then it belongs to  $p^{om} = p_{\mathcal{E}}$  and so  $\mathcal{E}(w) > 0$ , which is not our case.

So assume to have  $\mathcal{E} \in T_+^{\vee}$  such that  $p_{\mathcal{E}}$  is a regular point. Set  $W = \langle \text{Ker } \underline{\mathcal{E}} \cap T_+ \rangle_{\mathbb{Z}}$  and  $T'_+ = T_+ + W$ . Note that  $\text{Spec } k[T'_+]$  is an open subset of  $\text{Spec } k[T_+]$  that contains  $p_{\mathcal{E}}$ . Moreover  $k[T'_+]/p_{\mathcal{E}} = k[W]$ . Let  $v_1, \dots, v_q \in T_+$  be elements such that

$$T'_+ = \langle v_1, \dots, v_q \rangle_{\mathbb{N}} + W \quad \text{and} \quad \mathcal{E}(v_i) > 0$$

with  $q$  minimal. We claim that  $M = p_{\mathcal{E}}/p_{\mathcal{E}}^2 \simeq k[W]^q$ , where  $p_{\mathcal{E}}$  is thought in  $k[T'_+]$ . Indeed  $M$  is a  $k$ -vector space over the  $x_v$ ,  $v \in T'_+$  that satisfies:  $\mathcal{E}(v) > 0$  and whenever we have  $v = v' + v''$  with  $v', v'' \in T'_+$  it follows that  $\mathcal{E}(v') = 0$  or  $\mathcal{E}(v'') = 0$ . A simple computation shows that such a  $v$  must be of the form  $v_i + W$  for some  $i$ . But since we have chosen  $q$  minimal we have  $(v_i + W) \cap (v_j + W) = \emptyset$  if  $i \neq j$ . This implies that  $M$  is a free  $k[W]$ -module with basis  $x_{v_1}, \dots, x_{v_q}$ . This shows that  $q = \text{ht } p_{\mathcal{E}}$ .

Now set  $V = \langle v_1, \dots, v_q \rangle_{\mathbb{Z}}$ . Since  $V + W = T$ ,  $\text{rk } V \leq q$  and

$$k[W] \simeq k[T'_+]/p_{\mathcal{E}} \implies \text{rk } T = \dim k[T'_+] = \text{ht } p_{\mathcal{E}} + \dim k[W] = q + \text{rk } W$$

we obtain that  $v_1, \dots, v_q$  are independents. Let  $\mathcal{E}^1, \dots, \mathcal{E}^q$  given by  $\mathcal{E}^i(v_j) = \delta_{i,j}$  and  $\mathcal{E}^i|_W = 0$ . In particular  $W = \text{Ker } \underline{\mathcal{E}}$  and it is generated by elements in  $T_+$ . Since  $\mathcal{E}|_W = 0$  we have

$$\mathcal{E} = \sum_{i=1}^q \mathcal{E}(v_i) \mathcal{E}^i \quad \mathcal{E}(v_i) > 0$$

Moreover since  $T_+ \subseteq T'_+$  and  $\mathcal{E}^i \in T_+^{\vee}$  we get that  $\mathcal{E}^i \in T_+^{\vee}$ , as required.  $\square$

**Theorem 2.48.** *If  $\underline{\mathcal{E}}$  is a sequence of distinct indecomposable rays containing the smooth integral extremal rays then  $\pi_{\underline{\mathcal{E}}}$  induces an equivalence*

$$\left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}} \left| \begin{array}{l} V(z_{i_1}) \cap \dots \cap V(z_{i_s}) = \emptyset \\ \text{if } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \text{ is not a} \\ \text{smooth sequence} \end{array} \right. \right\} = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{Z}_{\phi}^{\text{sm}}) \xrightarrow{\simeq} \mathcal{Z}_{\phi}^{\text{sm}}$$

*Proof.* 2.47 says us that  $\mathcal{Z}_{\phi}^{\text{sm}} = \mathcal{X}_{\phi}^{\Theta}$ , where  $\Theta$  is the collection of all smooth sequences, while 2.40 allows us to replace  $\underline{\mathcal{E}}$  with the sequence of all smooth extremal rays. Therefore it is enough to apply 2.45 and 2.46.  $\square$

**Proposition 2.49.** *Let  $a: T_+ \rightarrow k \in \mathcal{X}_{\phi}(k)$ , where  $k$  is a field. Then  $a$  lies in  $\mathcal{Z}_{\phi}^{\text{sm}}$  if and only if there exists a smooth ray  $\mathcal{E} \in T_+^{\vee}$  and  $\lambda: T \rightarrow \bar{k}^*$  such that  $a = \lambda 0^{\mathcal{E}}$ .*

*Proof.* Apply 2.48 and 2.42.  $\square$

**2.4. Extension of objects from codimension 1.** In this subsection we want to explain how it is possible, in certain cases, to check that an object of  $\mathcal{X}_{\phi}$  over a 'good' scheme  $X$  comes (uniquely) from  $\mathcal{F}_{\underline{\mathcal{E}}}$  only checking what happens in codimension 1.

*Notation 2.50.* Given a scheme  $X$  we will denote by  $\underline{\text{Pic}} X$  the category whose objects are invertible sheaves and whose arrows are maps between them.

**Proposition 2.51.** *Let  $X \xrightarrow{f} Y$  be a map of scheme. If  $\underline{\mathrm{Pic}} Y \xrightarrow{f^*} \underline{\mathrm{Pic}} X$  is fully faithful (an equivalence) then also  $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$  is so.*

*Proof.* Let  $(\underline{\mathcal{L}}, a), (\underline{\mathcal{L}}', a') \in \mathcal{X}_\phi(Y)$  and  $\underline{\sigma}: f^*(\underline{\mathcal{L}}, a) \rightarrow f^*(\underline{\mathcal{L}}', a')$  be a map in  $\mathcal{X}_\phi(X)$ . Any map  $\sigma_i: f^*\mathcal{L}_i \rightarrow f^*\mathcal{L}'_i$  comes from a unique map  $\tau_i: \mathcal{L}_i \rightarrow \mathcal{L}'_i$ , i.e.  $\sigma_i = f^*\tau_i$ . Since

$$f^*(\underline{\tau}^{\phi(t)}(a(t))) = \underline{\sigma}^{\phi(t)}(f^*a(t)) = f^*(a'(t)) \implies \underline{\tau}^{\phi(t)}(a(t)) = a'(t)$$

$\underline{\tau}$  is a map  $(\underline{\mathcal{L}}, a) \rightarrow (\underline{\mathcal{L}}', a')$  such that  $f^*\underline{\tau} = \underline{\sigma}$ . We can conclude that  $f^*: \mathcal{X}_\phi(Y) \rightarrow \mathcal{X}_\phi(X)$  is fully faithful.

Now assume that  $\underline{\mathrm{Pic}} Y \xrightarrow{f^*} \underline{\mathrm{Pic}} X$  is an equivalence. We have to prove that  $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$  is essentially surjective. So let  $(\underline{\mathcal{M}}, b) \in \mathcal{X}_\phi(X)$ . Since  $f^*$  is an equivalence we can assume  $\mathcal{M}_i = f^*\mathcal{L}_i$  for some invertible sheaf  $\mathcal{L}_i$  on  $Y$ . Since for any invertible sheaf  $\mathcal{L}$  on  $Y$  one has that  $\mathcal{L}(Y) \simeq (f^*\mathcal{L})(X)$ , any section  $b(t) \in \underline{\mathcal{M}}^{\phi(t)}$  extends to a unique section  $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$ . Since

$$f^*(a(t) \otimes a(s)) = b(t) \otimes b(s) = b(t+s) = f^*(a(t+s)) \implies a(t) \otimes a(s) = a(t+s)$$

for any  $t, s \in T_+$  and  $a(0) = 1$ , it follows that  $(\underline{\mathcal{L}}, a) \in \mathcal{X}_\phi(Y)$  and  $f^*(\underline{\mathcal{L}}, a) = (\underline{\mathcal{M}}, b)$ .  $\square$

**Corollary 2.52.** *Let  $X \xrightarrow{f} Y$  be a map of schemes and consider a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{F}_{\underline{\mathcal{E}}} \\ f \downarrow & \nearrow & \downarrow \pi_{\underline{\mathcal{E}}} \\ Y & \longrightarrow & \mathcal{X}_\phi \end{array}$$

where  $\underline{\mathcal{E}}$  is a sequence of elements of  $T_+^\vee$ . Then if  $\underline{\mathrm{Pic}} X \xrightarrow{f^*} \underline{\mathrm{Pic}} Y$  is fully faithful (an equivalence) the dashed lifting is unique (exists).

*Proof.* It is enough to consider the 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\underline{\mathcal{E}}}(Y) & \xrightarrow{f^*} & \mathcal{F}_{\underline{\mathcal{E}}}(X) \\ \pi_{\underline{\mathcal{E}}} \downarrow & & \downarrow \pi_{\underline{\mathcal{E}}} \\ \mathcal{X}_\phi(Y) & \xrightarrow{f^*} & \mathcal{X}_\phi(X) \end{array}$$

and note that  $f^*$  is fully faithful (an equivalence) in both cases.  $\square$

**Theorem 2.53.** *Let  $X$  be a locally noetherian and locally factorial scheme,  $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$  be a sequence of distinct smooth integral rays and  $\Theta$  be a collection of smooth sequences with rays in  $\underline{\mathcal{E}}$ . Set*

$$\mathcal{C}_X^\Theta = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \begin{array}{l} \text{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \text{if } \nexists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\}$$

and

$$\mathcal{D}_X^\Theta = \left\{ \chi \in \mathcal{X}_\phi(X) \mid \forall p \in X \text{ with } \text{codim}_p X \leq 1 \right. \\ \left. \chi|_{\overline{k(p)}} \in \mathcal{X}_\phi^\Theta \right\}$$

Then  $\pi_{\underline{\mathcal{E}}}$  induces an equivalence of categories

$$\mathcal{C}_X^\Theta = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}_X^\Theta) \xrightarrow{\simeq} \mathcal{D}_X^\Theta$$

*Proof.* We claim that

$$\begin{aligned} \mathcal{C}_X^\Theta &= \{\chi \in \mathcal{F}_\Xi(X) \mid \exists U \subseteq X \text{ open subset s.t. } \text{codim}_X X - U \geq 2, \chi|_U \in \mathcal{F}_\Xi^\Theta(U)\} \\ &\subseteq \text{Taking into account the definition of } \mathcal{F}_\Xi^\Theta \text{ in 2.43, it is enough to consider} \end{aligned}$$

$$U = X - \bigcup_{\substack{\# \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta}}} V(z_{i_1}) \cap \dots \cap V(z_{i_s})$$

$\supseteq$  If  $p \in V(z_{i_1}) \cap \dots \cap V(z_{i_s})$  and  $\text{codim}_p X \leq 1$  then  $p \in U$  and again by definition of  $\mathcal{F}_\Xi^\Theta$  there exists  $\underline{\delta} \in \Theta$  such that  $\mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta}$ .

We also claim that

$$\begin{aligned} \mathcal{D}_X^\Theta &= \{\chi \in \mathcal{X}_\phi(X) \mid \exists U \subseteq X \text{ open subset s.t. } \text{codim}_X X - U \geq 2, \chi|_U \in \mathcal{X}_\phi^\Theta(U)\} \\ &\supseteq \text{Such a } U \text{ contains all the codimension 1 or 0 points of } X. \end{aligned}$$

$\subseteq$  Let  $\chi \in \mathcal{D}_X^\Theta$  and  $X \xrightarrow{g} \mathcal{X}_\phi$  be the induced map. If  $\xi$  is a generic point of  $X$ , we know that  $f(\xi) \in |\mathcal{X}_\phi^\Theta| \subseteq |\mathcal{Z}_\phi|$ . In particular  $f(|X|) \subseteq |\mathcal{Z}_\phi|$ . Since both  $X$  and  $\mathcal{Z}_\phi$  are reduced  $g$  factors to a map  $X \xrightarrow{g} \mathcal{Z}_\phi$ . Since  $\mathcal{X}_\phi^\Theta$  is an open substack of  $\mathcal{Z}_\phi$ , it follows that  $U = g^{-1}(\mathcal{X}_\phi^\Theta)$  is an open subscheme of  $X$ ,  $\chi|_U \in \mathcal{X}_\phi^\Theta(U)$  and, by definition of  $\mathcal{D}_X^\Theta$ ,  $\text{codim}_X X - U \geq 2$ .

Taking into account 2.45 it is clear that  $\mathcal{C}_X^\Theta = \pi_\Xi^{-1}(\mathcal{D}_X^\Theta)$ . We will make use of the fact that if  $U \subseteq X$  is an open subscheme such that  $\text{codim}_X X - U \geq 2$  then the restriction yields an equivalence  $\text{Pic } X \simeq \text{Pic } U$ . The map  $\mathcal{C}_X^\Theta \rightarrow \mathcal{D}_X^\Theta$  is essentially surjective since, given an object of  $\mathcal{D}_X^\Theta$ , the associated map  $X \xrightarrow{g} \mathcal{X}_\phi$  fits in a 2-commutative diagram

$$\begin{array}{ccc} U & \rightarrow & \mathcal{F}_\Xi^\Theta \subseteq \mathcal{F}_\Xi \\ \downarrow & & \downarrow \pi_\Xi \\ X & \xrightarrow{g} & \mathcal{X}_\phi \end{array}$$

and so lifts to a map  $X \rightarrow \mathcal{F}_\Xi$  thanks to 2.52.

It remains to show that  $\mathcal{C}_X^\Theta \rightarrow \mathcal{D}_X^\Theta$  is fully faithful. Let  $\chi, \chi' \in \mathcal{C}_X^\Theta$  and  $U, U'$  be the open subscheme given in the definition of  $\mathcal{C}_X^\Theta$ . Set  $V = U \cap U'$ . Taking into account 2.51 and 2.45 we have

$$\begin{array}{ccc} \text{Hom}_{\mathcal{F}_\Xi(X)}(\chi, \chi') & \longrightarrow & \text{Hom}_{\mathcal{X}_\phi(X)}(\chi, \chi') \\ \wr & & \wr \\ \text{Hom}_{\mathcal{F}_\Xi(V)}(\chi|_V, \chi'|_V) & \longrightarrow & \text{Hom}_{\mathcal{X}_\phi(V)}(\chi|_V, \chi'|_V) \\ \wr & & \wr \\ \text{Hom}_{\mathcal{F}_\Xi^\Theta(V)}(\chi|_V, \chi'|_V) & \xrightarrow{\simeq} & \text{Hom}_{\mathcal{X}_\phi^\Theta(V)}(\chi|_V, \chi'|_V) \end{array}$$

□

### 3. GALOIS COVERS FOR A DIAGONALIZABLE GROUP

In this section we will fix a finite diagonalizable group scheme  $G$  over  $\mathbb{Z}$  and we will call  $M = \text{Hom}(G, \mathbb{G}_m)$  its dual group. So  $M$  is a finite abelian group and  $G = D(M)$ . With abuse of notation we will write  $\mathcal{O}_U[M] = \mathcal{O}_U[G_U]$  and  $\mathcal{Z}_M = \mathcal{Z}_{D(M)}$ , the main component of  $D(M)\text{-Cov}$ . It turns out that in this case  $D(M)\text{-covers}$  have a nice and more explicit description.

In the first subsection we will show that  $D(M)\text{-Cov} \simeq \mathcal{X}_\phi$  for an explicit map  $T_+ \xrightarrow{\phi} \mathbb{Z}^M / \langle e_0 \rangle$  and that this isomorphism preserves the main irreducible components of both stacks. Moreover we will study the connection between  $D(M)\text{-Cov}$  and the equivariant Hilbert schemes  $\text{M-Hilb}^{\underline{m}}$  and prove some results about their geometry.

Then we will introduce an upper semicontinuous map  $|\mathrm{D}(M)\text{-Cov}| \xrightarrow{h} \mathbb{N}$  that yields a stratification by open substacks of  $\mathrm{D}(M)\text{-Cov}$ . We will also see that  $\{h = 0\}$  coincides with the open substack of  $\mathrm{D}(M)$ -torsors, while  $\{h \leq 1\}$  lies in the smooth locus of  $\mathcal{Z}_M$  and can be described by a particular set of smooth integral extremal rays. This will allow to describe the  $\mathrm{D}(M)$ -covers over locally noetherian and locally factorial scheme  $X$  with  $(\mathrm{char} X, |M|) = 1$  whose total space is regular in codimension 1 (which, a posteriori, is equivalent to the normal condition).

**3.1. The stack  $\mathrm{D}(M)\text{-Cov}$  and its main irreducible component  $\mathcal{Z}_M$ .** Consider a scheme  $U$  and a cover  $X = \mathrm{Spec} \mathcal{A}$  on it. An action of  $\mathrm{D}(M)$  on it consists of a decomposition

$$\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$$

such that  $\mathcal{O}_U \subseteq \mathcal{A}_0$  and the multiplication satisfies the rules  $\mathcal{A}_m \otimes \mathcal{A}_n \rightarrow \mathcal{A}_{m+n}$ . If  $X/U$  is a  $\mathrm{D}(M)$ -cover there exists an fppf covering  $\{U_i \rightarrow U\}$  such that  $\mathcal{A}|_{U_i} \simeq \mathcal{O}_{U_i}[M]$  as  $\mathrm{D}(M)$ -comodule. This means that for any  $m \in M$  we have

$$\forall i \ (\mathcal{A}_m)|_{U_i} \simeq \mathcal{O}_{U_i} \implies \mathcal{A}_m \text{ invertible}$$

Conversely any  $M$ -graded quasi-coherent algebra  $\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$  with  $\mathcal{A}_0 = \mathcal{O}_U$  and  $\mathcal{A}_m$  invertible for any  $m$  yields a  $\mathrm{D}(M)$ -cover  $\mathrm{Spec} \mathcal{A}$ .

So the stack  $\mathrm{D}(M)\text{-Cov}$  can be described as follows. An object of  $\mathrm{D}(M)\text{-Cov}(U)$  is given by a collection of invertible sheaves  $\mathcal{L}_m$  for  $m \in M$  with maps

$$\psi_{m,n}: \mathcal{L}_m \otimes \mathcal{L}_n \rightarrow \mathcal{L}_{m+n}$$

and an isomorphism  $\mathcal{O}_U \simeq \mathcal{L}_0$  satisfying the following relations:

$$\begin{array}{ccc} \text{Commutativity} & & \text{Associativity} \\ \begin{array}{ccc} \mathcal{L}_m \otimes \mathcal{L}_n & \xrightarrow{\simeq} & \mathcal{L}_n \otimes \mathcal{L}_m \\ & \searrow \psi_{m,n} \quad \swarrow \psi_{n,m} & \\ & \mathcal{L}_{m+n} & \end{array} & & \begin{array}{ccc} \mathcal{L}_m \otimes \mathcal{L}_n \otimes \mathcal{L}_t & \xrightarrow{\mathrm{id} \otimes \psi_{n,t}} & \mathcal{L}_m \otimes \mathcal{L}_{n+t} \\ \psi_{m,n} \otimes \mathrm{id} \downarrow & & \downarrow \psi_{m,n+t} \\ \mathcal{L}_{m+n} \otimes \mathcal{L}_t & \xrightarrow{\psi_{m+n,t}} & \mathcal{L}_{m+n+t} \end{array} \\ \\ \text{Neutral Element} & \mathcal{L}_m \xrightarrow{\simeq} \mathcal{L}_m \otimes \mathcal{O}_U \xrightarrow{\simeq} \mathcal{L}_m \otimes \mathcal{L}_0 \xrightarrow{\psi_{m,0}} \mathcal{L}_m & \\ & \text{id} & \end{array}$$

If we assume that  $\mathcal{L}_m = \mathcal{O}_U v_m$ , i.e. to have sections  $v_m$  generating  $\mathcal{L}_m$ , the maps  $\psi_{m,n}$  can be thought as elements of  $\mathcal{O}_U$  and the algebra structure is given by  $v_m v_n = \psi_{m,n} v_{m+n}$ . In this case we can rewrite the above conditions obtaining

$$(3.1) \quad \psi_{m,n} = \psi_{n,m}, \quad \psi_{m,0} = 1, \quad \psi_{m,n} \psi_{m+n,t} = \psi_{n,t} \psi_{m+t,m}$$

The functor that associates to a scheme  $U$  the functions  $\psi: M \times M \rightarrow \mathcal{O}_U$  satisfying the above conditions is clearly representable by the spectrum of the ring

$$(3.2) \quad R_M = \mathbb{Z}[x_{m,n}] / (x_{m,n} - x_{n,m}, x_{m,0} - 1, x_{m,n} x_{m+n,t} - x_{n,t} x_{m+t,m})$$

In this way we obtain a Zariski epimorphism  $\mathrm{Spec} R_M \rightarrow \mathrm{D}(M)\text{-Cov}$ , that we will prove to be smooth. We now want to prove that the stack  $\mathrm{D}(M)\text{-Cov}$  is isomorphic to a stack of the form  $\mathcal{X}_\phi$ .

**Definition 3.1.** Define  $\tilde{K}_+$  as the monoid quotient of  $\mathbb{N}^{M \times M}$  by the equivalence relation generated by

$$e_{m,n} \sim e_{n,m}, \quad e_{m,0} \sim 0, \quad e_{m,n} + e_{m+n,t} \sim e_{n,t} + e_{m+t,m}$$

Also define  $\phi_M: \tilde{K}_+ \rightarrow \mathbb{Z}^M / \langle e_0 \rangle$  by  $\phi_M(e_{m,n}) = e_m + e_n - e_{m+n}$ .

**Proposition 3.2.**  $R_M = \mathbb{Z}[\tilde{K}_+]$  and there exists an isomorphism

$$(3.3) \quad \mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov}$$

such that  $\mathrm{Spec} R_M = \mathrm{Spec} \mathbb{Z}[\tilde{K}_+] \longrightarrow \mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov}$  is the forgetful map. In particular

$$\mathrm{D}(M)\text{-Cov} \simeq [\mathrm{Spec} R_M / \mathrm{D}(\mathbb{Z}^M / \langle e_0 \rangle)]$$

*Proof.* The required isomorphism sends  $(\underline{\mathcal{L}}, \tilde{K}_+ \xrightarrow{\psi} \mathrm{Sym}^* \underline{\mathcal{L}}) \in \mathcal{X}_{\phi_M}$  to the object of  $\mathrm{D}(M)\text{-Cov}$  given by invertible sheaves  $(\mathcal{L}'_m = \mathcal{L}_m^{-1})$  and  $\psi_{m,n} = \psi(e_{m,n})$ .  $\square$

We want to prove that the isomorphism 3.3 sends  $\mathcal{Z}_{\phi_M}$  to  $\mathcal{Z}_M$  (see def. 1.3) and  $\mathcal{B}_{\phi_M}$  to  $\mathrm{BD}(M)$ . We need the following classical result on the structure of a  $\mathrm{D}(M)$ -torsor (see [SGA3Exp8, Proposition 4.1 and 4.6]):

**Proposition 3.3.** *Let  $M$  be a finite abelian group and  $P \longrightarrow U$  a  $\mathrm{D}(M)$ -equivariant map. Then  $P$  is an fppf  $\mathrm{D}(M)$ -torsor if and only if  $P \in \mathrm{D}(M)\text{-Cov}(U)$  and all the multiplication maps are isomorphisms.*

Now consider the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \rightarrow & \mathbb{Z}^M / \langle e_0 \rangle & \rightarrow & M \longrightarrow 0 \\ & & & & e_m & \longmapsto & m \end{array}$$

**Definition 3.4.** For  $m, n \in M$  we define

$$v_{m,n} = \phi_M(e_{m,n}) = e_m + e_n - e_{m+n} \in K$$

and  $K_+$  as the submonoid of  $K$  generated by the  $v_{m,n}$ . We will set  $x_{m,n} = x^{v_{m,n}} \in \mathbb{Z}[K_+]$  and, for  $\mathcal{E} \in K_+^\vee$ ,  $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$ .

**Lemma 3.5.** *The map*

$$\begin{array}{ccc} \tilde{K}_+ & \longrightarrow & K \\ e_{m,n} & \longmapsto & v_{m,n} \end{array}$$

is the associated group of  $\tilde{K}_+$  and  $K_+$  is its associated integral monoid. In particular we have a 2-cartesian diagram

$$\begin{array}{ccccc} \mathrm{Spec} \mathbb{Z}[K] & \longrightarrow & \mathrm{Spec} \mathbb{Z}[K_+] & \longrightarrow & \mathrm{Spec} R_M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BD}(M) & \longrightarrow & \mathcal{Z}_M & \longrightarrow & \mathrm{D}(M)\text{-Cov} \end{array}$$

*Proof.* Set  $x = \prod_{m,n} x_{m,n}$ . Since an object  $\psi \in \mathrm{Spec} R_M(U)$  is a torsor if and only if  $\psi_{m,n} \in \mathcal{O}_U^*$  for any  $m, n$ , it follows that  $(\mathrm{Spec} R_M)_x = \mathrm{BD}(M) \times_{\mathrm{D}(M)\text{-Cov}} \mathrm{Spec} R_M$ . We want to define an inverse to  $(R_M)_x \longrightarrow \mathbb{Z}[K]$ . If  $S_M$  is the universal algebra over  $R_M$  and we call  $w_m$  a graded basis of  $S_M$  with  $w_0 = 1$ ,  $(S_M)_x$  is a  $\mathrm{D}(M)$ -torsor over  $(R_M)_x$  and so  $w_m \in (S_M)_x^*$  for any  $m$ . In particular we can define a group homomorphism

$$\begin{array}{ccc} \mathbb{Z}^M / \langle e_0 \rangle & \longrightarrow & (S_M)_x^* \\ e_m & \longmapsto & w_m \end{array}$$

which restricts to a map  $K \longrightarrow (R_M)_x$  that sends  $v_{m,n}$  to  $x_{m,n}$ . In particular the map  $\tilde{K}_+ \longrightarrow K$  defined in the statement gives the associated group of  $\tilde{K}_+$  and has as image exactly  $K_+$ , which means that  $K_+$  is the integral monoid associated to  $\tilde{K}_+$ .  $\square$

In order to conclude the proof it's enough to apply 2.9 and 2.10.  $\square$

**Corollary 3.6.** *The isomorphism  $\mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov}$  (3.3) induces isomorphisms  $\mathcal{B}_{\phi_M} \simeq \mathrm{BD}(M)$  and  $\mathcal{Z}_{\phi_M} \simeq \mathcal{Z}_M$ . In particular  $\mathcal{Z}_M$  is an irreducible component of  $\mathrm{D}(M)\text{-Cov}$  and*

$$\mathrm{BD}(M) \simeq [\mathrm{Spec} \mathbb{Z}[K]/\mathrm{D}(\mathbb{Z}^M / \langle e_0 \rangle)] \text{ and } \mathcal{Z}_M \simeq [\mathrm{Spec} \mathbb{Z}[K_+]/\mathrm{D}(\mathbb{Z}^M / \langle e_0 \rangle)]$$

Note that the induced map  $\phi_M: K \rightarrow \mathbb{Z}^M / \langle e_0 \rangle$  is just the inclusion and so it is injective. This means that any result obtained in section 2 applies naturally in the context of  $\mathrm{D}(M)$ -covers. In particular now we show how we can describe the objects of  $\mathcal{F}_{\underline{\mathcal{E}}}$ , for a sequence of rays in  $\tilde{K}_+^\vee$ , in a simpler way.

**Proposition 3.7.** *Let  $M \simeq \prod_{i=1}^n \mathbb{Z}/l_i \mathbb{Z}$  be a decomposition and let  $m_1, \dots, m_n$  be the associated generators. Given  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$  define  $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$  as the stack whose objects over a scheme  $X$  are sequences  $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_n, \underline{\mathcal{M}} = \mathcal{M}_1, \dots, \mathcal{M}_r, \underline{z} = z_1, \dots, z_r, \underline{\mu} = \mu_1, \dots, \mu_n$  where  $\underline{\mathcal{L}}, \underline{\mathcal{M}}$  are invertible sheaves over  $X$ ,  $z_i \in \mathcal{M}_i$  and  $\underline{\mu}$  are isomorphisms*

$$\mu_i: \mathcal{L}_i^{-l_i} \xrightarrow{\simeq} \underline{\mathcal{M}}^{\mathcal{E}(l_i e_{m_i})} = \mathcal{M}_1^{\mathcal{E}^1(l_i e_{m_i})} \otimes \dots \otimes \mathcal{M}_r^{\mathcal{E}^r(l_i e_{m_i})}$$

Then we have an isomorphism of stacks

$$\begin{aligned} \mathcal{F}_{\underline{\mathcal{E}}} &\longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}} \\ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) &\longmapsto ((\mathcal{L}_{m_i})_{i=1, \dots, n}, \underline{\mathcal{M}}, \underline{z}, (\lambda(l_i e_{m_i}))_{i=1, \dots, n}) \end{aligned}$$

*Proof.* We want to find  $\sigma, V, v_1, \dots, v_q$  as in 2.26 such that  $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}, \sigma} = \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$  and that the map in the statement coincide with the one defined in 2.28. Set  $\delta^i: M \rightarrow \{0, \dots, l_i - 1\}$  as the map such that  $\pi_i(m) = \pi_i(\delta_m^i m_i)$ , where  $\pi_i: M \rightarrow \mathbb{Z}/l_i \mathbb{Z}$ , and think it also as a map  $\delta^i: \mathbb{Z}^M / \langle e_0 \rangle \rightarrow \mathbb{Z}$ . Set  $V = \bigoplus_{i=1}^n \mathbb{Z} e_{m_i}$ ,  $v_i = e_{m_i}$  and  $\sigma: \mathbb{Z}^M / \langle e_0 \rangle \rightarrow V$  as  $\sigma(e_m) = \sum_{i=1}^n \delta_m^i v_i$ . Clearly  $(\mathrm{id} - \sigma)\mathbb{Z}^M / \langle e_0 \rangle \subseteq K$  and  $(\mathrm{id} - \sigma)V = 0$ . So  $W = \sigma K$ . We have

$$\sigma(v_{m,n}) = \sum_{i=1}^n \delta_{m,n}^i v_i \in \bigoplus_{i=1}^n l_i \mathbb{Z} v_i$$

since for any  $i$   $\delta_{m,n}^i \in \{0, l_i\}$ . On the other hand  $\sigma(v_{(l_i-1)m_i, m_i}) = l_i v_i$ . Therefore we have  $W = \bigoplus_{i=1}^n l_i \mathbb{Z} v_i$ . It's now easy to check that all the definitions agree.  $\square$

We now want to express the relation between  $\mathrm{D}(M)\text{-Cov}$  and the equivariant Hilbert scheme, that can be defined as follows. Given  $\underline{m} = m_1, \dots, m_r \in M$ , so that  $\mathrm{D}(M)$  acts on  $\mathbb{A}_{\mathbb{Z}}^r = \mathrm{Spec} \mathbb{Z}[x_1, \dots, x_r]$  with graduation  $\deg x_i = m_i$ , we define  $\mathrm{M}\text{-Hilb}^{\underline{m}}: \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathrm{set}$  as the functor that associates to a scheme  $Y$  the pairs  $(X \xrightarrow{f} Y, j)$  where  $X \in \mathrm{D}(M)\text{-Cov}(Y)$  and  $j: X \rightarrow \mathbb{A}_Y^r$  is an equivariant closed immersion over  $Y$ . Such a pair can be also thought as a coherent sheaf of algebras  $\mathcal{A} \in \mathrm{D}(M)\text{-Cov}(Y)$  together with a graded surjective map  $\mathcal{O}_Y[x_1, \dots, x_r] \rightarrow \mathcal{A}$ . This functor is proved to be a scheme of finite type in [HS02].

**Proposition 3.8.** *Let  $\underline{m} = m_1, \dots, m_r \in M$ . The forgetful map  $\vartheta_{\underline{m}}: \mathrm{M}\text{-Hilb}^{\underline{m}} \rightarrow \mathrm{D}(M)\text{-Cov}$  is a smooth zariski epimorphism onto the open substack  $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$  of  $\mathrm{D}(M)\text{-Cov}$  of sheaves of algebras  $\mathcal{A}$  such that, for any  $y \in Y$ ,  $\mathcal{A} \otimes k(y)$  is generated in the degrees  $m_1, \dots, m_r$  as a  $k(y)$ -algebra. Moreover  $\mathrm{M}\text{-Hilb}^{\underline{m}}$  is an open subscheme of a vector bundle over  $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$ .*

*Proof.* Let  $\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m \in \mathrm{D}(M)\text{-Cov}$  and consider the map

$$\eta_{\mathcal{A}}: \mathrm{Sym}(\mathcal{A}_{m_1} \oplus \dots \oplus \mathcal{A}_{m_r}) \rightarrow \mathcal{A}$$



It's easy to check that  $\eta_{\mathcal{A}}$  is surjective if and only if  $\mathcal{A} \in \mathrm{D}(M)\text{-Cov}^{\underline{m}}$ . Therefore  $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$  is an open substack of  $\mathrm{D}(M)\text{-Cov}$  and clearly contains the image of  $\vartheta_{\underline{m}}$ . Consider now a cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & \mathrm{M}\text{-Hilb}^{\underline{m}} \\ \downarrow & & \downarrow \vartheta_{\underline{m}} \\ T & \xrightarrow{\mathcal{A}} & \mathrm{D}(M)\text{-Cov}^{\underline{m}} \end{array}$$

and let  $U \xrightarrow{\phi} T$  be a map.  $F(U)$  is given by a graded surjection  $\mathcal{O}_U[x_1, \dots, x_r] \rightarrow \mathcal{B}$  and an isomorphism  $\mathcal{B} \simeq \phi^* \mathcal{A}$ . This is equivalent to giving a graded surjection  $\mathcal{O}_U[x_1, \dots, x_r] \rightarrow \phi^* \mathcal{A}$ . In this way we obtain a map

$$F \xrightarrow{g_T} \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathcal{A}_{m_i}) \simeq \mathrm{Spec} \mathrm{Sym}(\bigoplus_i \mathcal{A}_{m_i}^{-1})$$

We claim that this is an open immersion. Indeed given  $(a_i)_i: U \rightarrow \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathcal{A}_{m_i})$ , the fiber product with  $F$  is the locus where the induced graded map  $\mathcal{O}_U[x_1, \dots, x_r] \rightarrow \mathcal{A} \otimes_{\mathcal{O}_U}$  is surjective, that is an open subscheme of  $U$ . In particular  $F$  is smooth and so  $\vartheta_{\underline{m}}$  is smooth too. It's easy to check that it is also a zariski epimorphism. Finally the vector bundle  $\mathcal{N}$  of the statement is defined over any  $U \rightarrow \mathrm{D}(M)\text{-Cov}^{\underline{m}}$  given by  $\mathcal{A} = \bigoplus_m \mathcal{A}_m$  by  $\mathcal{N}|_U = \bigoplus_i \mathcal{A}_{m_i}^{-1}$ .  $\square$

*Remark 3.9.* If the sequence  $\underline{m}$  contains any elements of  $M - \{0\}$ , then  $\mathrm{D}(M)\text{-Cov}^{\underline{m}} = \mathrm{D}(M)\text{-Cov}$ . Therefore in this case  $\mathrm{M}\text{-Hilb}^{\underline{m}}$  is an atlas for  $\mathrm{D}(M)\text{-Cov}$ .

*Remark 3.10.* We have cartesian diagrams

$$\begin{array}{ccccccc} W_{\underline{m}} & \longrightarrow & V_{\underline{m}} & \longrightarrow & U_{\underline{m}} & \longrightarrow & \mathrm{Spec} R_M \\ \downarrow \text{torsors} & & \downarrow \text{open} & & \downarrow \text{vector} & & \downarrow \text{open} \\ \mathrm{M}\text{-Hilb}^{\underline{m}} & \longrightarrow & H_{\underline{m}} & \longrightarrow & \mathrm{D}(M)\text{-Cov}^{\underline{m}} & \longrightarrow & \mathrm{D}(M)\text{-Cov} \end{array}$$

In particular, since  $\mathrm{BD}(M) \subseteq \mathrm{D}(M)\text{-Cov}^{\underline{m}}$ , we can conclude that  $\vartheta_{\underline{m}}^{-1}(\mathcal{Z}_M)$  is the main irreducible component of  $\mathrm{M}\text{-Hilb}^{\underline{m}}$ . Moreover the above diagram shows that  $\mathrm{M}\text{-Hilb}^{\underline{m}}$  and  $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$ , as well as their main irreducible components, share many properties like smoothness, connection, integrality, reducibility.

We want now study some geometrical properties of the stack  $\mathrm{D}(M)\text{-Cov}$  and, therefore, of the equivariant Hilbert schemes.

*Remark 3.11.* The ring  $R_M$  can be written as quotient of the ring  $\mathbb{Z}[x_{m,n}]_{(m,n) \in J}$ , where  $J$  is  $\{(m,n) \in M^2 \mid m, n, m+n \neq 0\}$  divided by the equivalence relation  $(m,n) \sim (n,m)$ , by the ideal

$$I = \left( \begin{array}{l} x_{m,n}x_{m+n,t} - x_{n,t}x_{n+t,m} \text{ with } m, n, t, m+n, n+t, m+n+t \neq 0 \text{ and } m \neq t, \\ x_{-m,t}x_{-m+t,m} - x_{-m,s}x_{-m+s,m} \text{ with } m, s, t \neq 0 \text{ and distinct} \end{array} \right)$$

Indeed the first relations are trivial when one of  $m, n, t$  is zero or  $m = t$ , while if  $m+n=0$  yield relations  $x_{m,-m} = x_{-m,t}x_{-m+t,m}$ . Using these last relations we can remove all the variable  $x_{m,n}$  with  $0 \in \{m, n, m+n\}$ .

*Remark 3.12.* There exist a map  $f: \tilde{K}_+ \rightarrow \mathbb{N}$  such that for any  $m, n \neq 0$  we have  $f(e_{m,n}) = 1$  if  $m+n \neq 0$ ,  $f(e_{m,-m}) = 2$  otherwise. In particular  $f(v) = 0$  only if  $v = 0$ . Moreover  $f$  induces an  $\mathbb{N}$ -graduation on both  $(R_M \otimes A)$  and  $\mathbb{Z}[K_+] \otimes A$ , where  $A$  is a ring, such that the degree zero part is  $A$  and that the elements  $x_{m,n}$  with  $m+n \neq 0$  are homogeneous of degree 1.  $f$  is obtained as composition  $\tilde{K}_+ \rightarrow K \subseteq \mathbb{Z}^M / \langle e_0 \rangle \xrightarrow{h} \mathbb{Z}$ , where  $h(e_m) = 1$  if  $m \neq 0$ .

One of the open problems in the theory of equivariant Hilbert schemes is whether those schemes are connected. As said above  $M\text{-Hilb}^{\underline{m}}$  is connected if and only if  $D(M)\text{-Cov}^{\underline{m}}$  is so. What we can say here is:

**Theorem 3.13.** *The stack  $D(M)\text{-Cov}$  is connected with geometrically connected fibers. If  $M - \{0\} \subseteq \underline{m}$ , then  $M\text{-Hilb}^{\underline{m}}$  has the same properties.*

*Proof.* It's enough to prove that  $\text{Spec } R_M \otimes k$  is connected for any field  $k$ . But  $R_M \otimes k$  has an  $\mathbb{N}$ -graduation such that  $(R_M \otimes k)_0 = k$  by 3.12 and it is a general fact that such an algebra doesn't contain non trivial idempotents.  $\square$

We now want to discuss the problem of the reducibility of  $D(M)\text{-Cov}$ .

**Definition 3.14.** Let  $X$  be a scheme over a base scheme  $S$ .  $X$  is said universally reducible over  $S$  if for any base change  $S' \rightarrow S$  the scheme  $X \times_S S'$  is reducible. A scheme is universally reducible if it is so over  $\mathbb{Z}$ .

*Remark 3.15.* It's easy to check that  $X$  is universally reducible over  $S$  if and only if all the fibers are reducible.

**Lemma 3.16.** *If there exist  $m, n, t, a \in M$  such that*

- (1)  $m, n, t$  are distinct and not zero;
- (2)  $a \neq 0, m, n, t, m-n, n-t, t-n, m-t, 2m-t, 2n-t, m+n-t, m+n-2t$ ;
- (3)  $2a \neq m+n-t$ ;

*then  $\text{Spec } R_M$  is universally reducible.*

*Proof.* Let  $k$  be a field and  $I = (\underline{x}^{\alpha_i} - \underline{x}^{\beta_i})$  be an ideal of  $k[x_1, \dots, x_r] = k[\underline{x}]$ . We will say that  $\alpha \in \mathbb{N}^r$  is transformable (with respect to  $I$ ) if there exists  $i$  such that  $\alpha_i \leq \alpha$  or  $\beta_i \leq \alpha$ . Here by  $\alpha \leq \beta \in \mathbb{N}^r$  we mean  $\alpha_j \leq \beta_j$  for all  $j$ . A direct computation shows that if  $\underline{x}^\alpha - \underline{x}^\beta \in I$  and  $\alpha \neq \beta$ , then both  $\alpha$  and  $\beta$  are transformable.

We will use the above notation for the ideal  $I$  defining  $R_M \otimes k$  as in 3.11. In particular the elements  $\alpha_i, \beta_i \in \mathbb{N}^J$  associated to the ideal  $I$  are of the form  $e_{u,v} + e_{u+v,w}$  with  $u, v, u+v, w, u+v+w \neq 0$ .

Set  $\mu = \prod_{m,n} x_{m,n}$ . Since  $R_M \otimes k \rightarrow k[K_+] \subseteq k[K] = (R_M \otimes k)_\mu$ , there exists  $N > 0$  such that  $P = \text{Ker}(R_M \otimes k \rightarrow k[K_+]) = \text{Ann } \mu^N$ . Our strategy will be to find an element of  $P$  which is not nilpotent. Since  $P$  is a minimal prime, being  $\text{Spec } k[K_+]$  an irreducible component of  $\text{Spec } R_M \otimes k$ , it follows that  $R_M \otimes k$  is reducible. Now consider  $\alpha = e_{a,m-a} + e_{m+n-t-a,t+a-m} + e_{t+a-n,n-a}$ ,  $\beta = e_{m+n-t-a,t+a-n} + e_{a,n-a} + e_{m-a,t+a-m} \in \mathbb{N}^J$  and  $z = \underline{x}^\alpha - \underline{x}^\beta$ . We will show that  $\mu z = 0$ , i.e.  $z \in P$  and that  $z$  is not nilpotent. First of all note that  $z$  is well defined since for any  $e_{u,v}$  in  $\alpha$  or  $\beta$  we have  $u, v \neq 0$  and  $0 \neq u+v \in \{m, n, t\}$  thanks to 1), 2). Let  $S_M$  be the universal algebra over  $R_M$ , i.e.  $S_M = \bigoplus_{m \in M} R_M v_m$  with  $v_m v_n = x_{m,n} v_{m+n}$  and  $v_0 = 1$ . By construction we have

$$\begin{aligned} (v_a v_{m-a})(v_{m+n-t-a} v_{t+a-m})(v_{t+a-n} v_{n-a}) &= \underline{x}^\alpha v_m v_n v_t = \\ (v_{m+n-t-a} v_{t+a-n})(v_a v_{n-a})(v_{m-a, t+a-m}) &= \underline{x}^\beta v_m v_n v_t \end{aligned}$$

So  $\underline{x}^\alpha x_{m,n} x_{m+n,t} v_{m+n+t} = \underline{x}^\beta x_{m,n} x_{m+n,t} v_{m+n+t}$  and therefore  $z\mu = 0$ , i.e.  $z \in P$ .

Now we want to prove that any linear combination  $\gamma = a\alpha + b\beta \in \mathbb{N}^J$  with  $a, b \in \mathbb{N}$  is not transformable. First remember that each  $e_{u,v}$  in  $\gamma$  is such that  $u+v \in \{m, n, t\}$ . If we will have  $e_{u,v} + e_{u+v,w} \leq \gamma$  then there must exist  $e_{i,j} \leq \gamma$  such that  $i \in \{m, n, t\}$  or  $j \in \{m, n, t\}$ . Condition 2) is exactly what we need to avoid this situation and can be written as  $\{a, m-a, m+n-t-a, t+a-m, t+a-n, n-a\} \cap \{m, n, t\} = \emptyset$ .

In particular, if we think  $\tilde{K}_+$  as a quotient of  $\mathbb{N}^J$ , we have that  $a\alpha + b\beta = a'\alpha + b'\beta$  in  $\tilde{K}_+$  if and only if they are equals in  $\mathbb{N}^J$ . Assume for a moment that  $\alpha \neq \beta$  in

$\mathbb{N}^J$ . Clearly this means that  $\alpha$  and  $\beta$  are  $\mathbb{Z}$ -independent in  $\mathbb{Z}^J$ . Since any linear combination of  $\alpha$  and  $\beta$  is not transformable, it follows that  $\underline{x}^\alpha, \underline{x}^\beta$  are algebraically independent over  $k$  in  $R_M \otimes k$  and, in particular, that  $z = \underline{x}^\alpha - \underline{x}^\beta$  cannot be nilpotent. So it remains to prove that  $\alpha \neq \beta$  in  $\mathbb{N}^J$ . Note that for any  $i \in \{m, n, t\}$  there exists only one  $e_{u,v}$  in  $\alpha$  such that  $u + v = i$  and the same happens for  $\beta$ . So, if  $\alpha = \beta$  and since  $m, n, t$  are distinct, those terms have to be equal, for instance  $e_{a, m-a} = e_{m+n-t-a, t+a-n}$ . But  $a \neq m + n - t - a$  by 3), while  $a \neq t + a - n$  since  $t \neq n$ . Therefore  $\alpha \neq \beta$ .  $\square$

**Corollary 3.17.** *If  $|M| > 7$  and  $M \not\simeq (\mathbb{Z}/2\mathbb{Z})^3$  then  $D(M)$ -Cov is universally reducible and the same holds for  $M\text{-Hilb}^{\underline{m}}$  provided that  $\underline{m}$  contains any element of  $M - \{0\}$ .*

*Proof.* We have to show that  $R_M$  is universally reducible and so we will apply 3.16. If  $M = C \times T$ , where  $C$  is cyclic with  $|C| \geq 4$  and  $T \neq 0$  we can choose:  $m$  a generator of  $C$ ,  $n = 3m$ ,  $t = 2m$  and  $a \in T - \{0\}$ . If  $M$  cannot be written as above, there are four remaining cases. 1)  $M \simeq \mathbb{Z}/8\mathbb{Z}$ : choose  $m = 2$ ,  $n = 4$ ,  $t = 6$ ,  $a = 1$ . 2)  $M$  cyclic with  $|M| > 8$  and  $|M| \neq 10$ : choose  $m = 1$ ,  $n = 2$ ,  $t = 3$ ,  $a = 5$ . 3)  $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$  with  $l \geq 4$ : choose  $m = e_1$ ,  $n = e_2$ ,  $t = e_3$ ,  $a = e_4$ . 4)  $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$  with  $l \geq 2$ : choose  $m = e_1$ ,  $n = 2e_1$ ,  $t = e_2$ ,  $a = m + t = e_1 + e_2$ .  $\square$

**Proposition 3.18.**  *$D(M)$ -Cov is smooth if and only if  $\mathcal{Z}_M$  is so. This happens if and only if  $M \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and in this case  $D(M)$ -Cov =  $\mathcal{Z}_M$ . To be more precise  $R_M = \mathbb{Z}[x_{m,n}]_{(m,n) \in J}$ , where  $J$  is the set defined in 3.11.*

*In particular  $M\text{-Hilb}^{\underline{m}}$  is smooth and irreducible for any sequence  $\underline{m}$  if  $M$  is as above. Otherwise, if  $M - \{0\} \subseteq \underline{m}$ ,  $M\text{-Hilb}^{\underline{m}}$  is not smooth.*

*Proof.* Let  $k$  be a field. Note that

$$D(M)\text{-Cov smooth} \iff R_M \text{ smooth} \implies \mathcal{Z}_M \text{ smooth} \implies k[K_+]/k \text{ smooth}$$

We first prove that if  $k[K_+]$  is smooth then  $M$  has to be one of the groups of the statement. We have  $K_+ \simeq \mathbb{N}^r \oplus \mathbb{Z}^s$  and therefore  $k[K_+]$  is UFD. We will consider  $k[K_+]$  endowed with the  $\mathbb{N}$ -graduation defined in 3.12. Since any of the  $x_{m,n}$  has degree 1, it is irreducible and so prime. If we have a relation  $x_{m,n}x_{m+n,t} = x_{n,t}x_{n+t,m}$  with  $m, n, t, m+n, n+t, m+n+t \neq 0$  and  $m \neq t$ , then  $x_{m,n} \mid x_{n,t}x_{n+t,m}$  implies that  $x_{m,n} = x_{n,t}$  or  $x_{m,n} = x_{n+t,m}$ , which is impossible thanks to our assumptions. We will prove that if  $M$  is not isomorphic to one of the group in the statement, then such a relation exists. Clearly it is enough to find this relation in a subgroup of  $M$ . So it is enough to consider the following cases. 1)  $M$  cyclic with  $|M| \geq 5$ : choose  $m = n = 1$ ,  $t = 2$ . 2)  $M \simeq \mathbb{Z}/4\mathbb{Z}$ : choose  $m = 1$ ,  $n = 2$ ,  $t = 3$ . 3)  $M \simeq (\mathbb{Z}/2\mathbb{Z})^3$ : choose  $m = e_1$ ,  $n = e_2$ ,  $t = e_3$ . 4)  $M \simeq (\mathbb{Z}/3\mathbb{Z})^2$ : choose  $m = n = e_1$ ,  $t = e_2$ .

We now want to prove that when  $M$  is as in the statement, then the ideal  $I$  of 3.11 is zero. If we have a relation as in the first row, since  $m \neq t$  we have  $|M| \geq 3$ . If  $M \simeq \mathbb{Z}/3\mathbb{Z}$  then  $t = 2m$  and  $m + t = 0$ . If  $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$ , if  $m, n, t$  are distinct then  $m + n + t = 0$ , otherwise  $m = n$  and  $m + n = 0$ . If we have a relation as in the second row, since  $m, t, s$  are distinct, we must have  $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$ . Therefore  $m + t = s$  and the relation become trivial.  $\square$

**Corollary 3.19.**  *$D(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ -Cov is isomorphic to the stack of sequences  $(\mathcal{L}_i, \psi_i)_{i=1,2,3}$ , where  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$  are invertible sheaves and  $\psi_1: \mathcal{L}_2 \otimes \mathcal{L}_3 \rightarrow \mathcal{L}_1$ ,  $\psi_2: \mathcal{L}_1 \otimes \mathcal{L}_3 \rightarrow \mathcal{L}_2$ ,  $\psi_3: \mathcal{L}_1 \otimes \mathcal{L}_2 \rightarrow \mathcal{L}_3$  are maps.*

*Proof.* Set  $M = (\mathbb{Z}/2\mathbb{Z})^2$ . Thanks to 3.18, we know that  $\tilde{K}_+ = K_+ \simeq \mathbb{N}v_{e_1, e_2} \oplus \mathbb{N}v_{e_1, e_1+e_2} \oplus \mathbb{N}v_{e_2, e_1+e_2}$ . So an object of  $D(M)$ -Cov is given by invertible sheaves

$\mathcal{L}_1 = \mathcal{L}_{e_1}$ ,  $\mathcal{L}_2 = \mathcal{L}_{e_2}$ ,  $\mathcal{L}_3 = \mathcal{L}_{e_1+e_2}$  and maps  $\psi_1 = \psi_{e_2, e_1+e_2}$ ,  $\psi_2 = \psi_{e_1, e_1+e_2}$ ,  $\psi_3 = \psi_{e_1, e_2}$ .  $\square$

*Remark 3.20.*  $D(\mathbb{Z}/4\mathbb{Z})\text{-Cov}$  and  $\mathbb{Z}/4\mathbb{Z}\text{-Hilb}^{\underline{m}}$ , for any sequence  $\underline{m}$ , are integral and normal since one can check directly that  $R_{\mathbb{Z}/4\mathbb{Z}} = \mathbb{Z}[x_{1,2}, x_{3,3}, x_{2,3}, x_{1,1}]/(x_{1,2}x_{3,3} - x_{2,3}x_{1,1})$ . I'm not able to prove that  $D(M)\text{-Cov}$  is irreducible when  $M$  is one of  $\mathbb{Z}/5\mathbb{Z}$ ,  $\mathbb{Z}/6\mathbb{Z}$ ,  $\mathbb{Z}/7\mathbb{Z}$ ,  $(\mathbb{Z}/2\mathbb{Z})^3$ . Anyway the first two cases seems to be integral thanks to a computer program, while for the last ones there are some techniques that can be used to study this problem but they are too complicated to be explained here.

**3.2. The invariant  $h$ :**  $|D(M)\text{-Cov}| \rightarrow \mathbb{N}$ . In this subsection we investigate the local structure of a  $D(M)$ -cover, especially over a local ring. In particular we will be able to define an upper semicontinuous map  $h: |D(M)\text{-Cov}| \rightarrow \mathbb{N}$ .

*Notation 3.21.* Given a ring  $A$ , we will write  $B \in \text{Spec } R_M(A)$  meaning that  $B$  is an  $M$ -graded  $A$ -algebra with a given  $M$ -graded basis, usually denoted by  $\{v_m\}_{m \in M}$  with  $v_0 = 1$ , and a given multiplication  $\psi$  such that

$$B = \bigoplus_{m \in M} Av_m, \quad v_m v_n = \psi_{m,n} v_{m+n}$$

**Lemma 3.22.** *Let  $A$  be a ring and  $B \in \text{Spec } R_M(A)$ , with graded basis  $v_m$  and multiplication map  $\psi$ . Then the set*

$$H_\psi = H_{B/A} = \{m \in M \mid v_m \in B^*\} = \{m \in M \mid \psi_{m,-m} \in A^*\}$$

*is a subgroup of  $M$ . Moreover if  $m, n \in M$  and  $h \in H_\psi$  then  $\psi_{m,n}$  and  $\psi_{m,n+h}$  differs by an element of  $A^*$ . If  $H < H_\psi$  then  $C = \bigoplus_{m \in H} Av_m \in \text{BD}(H)(A)$ . Moreover if  $\sigma: M/H \rightarrow M$  gives representatives of  $M/H$  in  $M$  and we set  $w_m = v_{\sigma(m)}$  for  $m \in M/H$  we have*

$$B = \bigoplus_{m \in M/H} Cw_m \in \text{Spec } R_{M/H}(C)$$

*Finally if we denote by  $\psi'$  the induced multiplication on  $B$  over  $C$  we have  $H_{\psi'} = H_\psi/H$  and for any  $m, n \in M$   $\psi'_{m,n}$  and  $\psi_{m,n}$  differ by an element of  $C^*$ .*

*Proof.* From the relations  $v_m v_{-m} = \psi_{m,-m}$ ,  $v_m^{|M|-1} = \lambda v_{-m}$ ,  $v_m^{|M|} = \lambda \psi_{m,-m}$  and  $v_m v_n = \psi_{m,n} v_{m+n}$  we see that  $v_m \in B^* \iff \psi_{m,-m} \in A^*$  and that  $H_\psi < M$ . From 3.1 we get the relations  $\psi_{-h,h} = \psi_{h,u} \psi_{h+u,-h}$  and  $\psi_{m,n} \psi_{m+n,h} = \psi_{n,h} \psi_{m,n+h}$ . So if  $h \in H$  then  $\psi_{h,u} \in A^*$  for any  $u$  and  $\psi_{m,n}$  and  $\psi_{m,n+h}$  differ by an element of  $A^*$ .

Now consider the second part of the statement. From 3.3 we know that  $C$  is a torsor over  $A$ . Since for any  $m$  we have  $v_m = (\psi_{h,m}/v_h) v_{\sigma(\overline{m})}$ , where  $h = \sigma(\overline{m}) - m \in H$  we obtain the writing of  $B$  as  $M/H$  graded  $C$ -algebra and that

$$\psi'_{m,n} = \psi_{\sigma(m), \sigma(n)} (\psi_{h, \sigma(m)+\sigma(n)}/v_h) \text{ where } h = \sigma(m+n) - \sigma(m) - \sigma(n)$$

From the above equation it is easy to conclude the proof.  $\square$

**Definition 3.23.** Given a ring  $A$  and  $B \in \text{Spec } R_M(A)$  we continue to use the notation  $H_{B/A}$  introduced in 3.22 and we will call the algebra  $C$  obtained in it setting  $H = H_{B/A}$  the *maximal torsor* of the extension  $B/A$ . If  $\mathcal{E} \in K_+^\vee$  we will write  $H_{\mathcal{E}} = H_{B/k}$  where  $B$  is the algebra induced by the multiplication  $0^{\mathcal{E}}$  and  $k$  is any field. In particular

$$H_{\mathcal{E}} = \{m \in M \mid \mathcal{E}_{m,-m} = 0\}$$

Finally if  $f: X \rightarrow Y \in D(M)\text{-Cov}(Y)$  and  $q \in Y$  we define  $\mathcal{H}_f(q) = H_{\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}}$ .

**Proposition 3.24.** *We have a map*

$$\begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{\mathcal{H}} & \{\text{subgroups of } M\} \\ B/k & \longmapsto & H_{B/k} \end{array}$$

such that, if  $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$  is given by  $X \xrightarrow{f} Y$ , then  $\mathcal{H}_f = \mathcal{H} \circ u$ .

*Proof.* It's enough to note that if  $A$  is a local ring,  $B \in \mathrm{D}(M)\text{-Cov}(A)$  is given by multiplications  $\psi$  and  $\pi: A \rightarrow A/m_A \rightarrow k$  is a morphism, where  $k$  is a field, then  $\psi_{m,-m} \in A^* \iff \pi(\psi_{m,-m}) \neq 0$ .  $\square$

**Remark 3.25.** Let  $(A, m_A)$  be a local ring and  $B \in \mathrm{Spec} R_M(A)$  with  $M$ -graded basis  $\{v_m\}_{m \in M}$ . Then  $H_{B/A} = \mathcal{H}_{B/A}(m_A)$ . If  $H_{B/A} = 0$  then  $B$  is local with maximal ideal

$$m_B = m_A \oplus \bigoplus_{m \in M - \{0\}} Av_m$$

and residue field  $B/m_B = A/m_A$ . In particular  $m_B/m_B^2$  is  $M$ -graded.

**Lemma 3.26.** *Let  $A$  be a local ring and  $B = \bigoplus_{m \in M} Av_m \in \mathrm{D}(M)\text{-Cov}(A)$  such that  $H_{B/A} = 0$ . If  $m_1, \dots, m_r \in M$  then  $B$  is generated in degrees  $m_1, \dots, m_r$  as an  $A$ -algebra if and only if  $m_B = (m_A, v_{m_1}, \dots, v_{m_r})_B$ .*

*Proof.* We can write  $m_B = m_A \oplus \bigoplus_{m \in M - \{0\}} Av_m$ . Denote  $\underline{v} = v_{m_1}, \dots, v_{m_r}$  and  $\pi(\alpha) = \sum_i \alpha_i m_i$  for  $\alpha \in \mathbb{N}^r$ . The only if follows since given  $l \in M - \{0\}$  there exists a relation of the form  $v_l = \mu \underline{v}^\alpha$  with  $\mu \in A^*$  and  $\alpha \neq 0$  and so  $v_l \in (m_A, v_{m_1}, \dots, v_{m_r})_B$ . For the converse note that, given  $l \in M - \{0\}$ ,  $v_l \in m_B = (m_A, v_{m_1}, \dots, v_{m_r})$  means that we have a relation  $v_l = \lambda v_{l'} v_{m_i}$  for some  $i$ ,  $\lambda \in A^*$  and  $l' = l - m_i$ . Moreover  $v_l \notin A[\underline{v}]$  implies that  $v_{l'} \notin A[\underline{v}]$  and  $l' \neq 0$ . If, by contradiction, we have such an element  $l$  we can write  $v_l = \mu v_{n_1} \cdots v_{n_s}$  with  $n_i \in M - \{0\}$  and  $s \geq |M|^2$ . In particular there must exist  $i$  such that  $m = n_i$  appears at least  $|M|$  times in this product. So  $m_A \ni v_m^{|M|} \mid v_l$  and  $v_l \in m_A B$ , which is not the case.  $\square$

Assume to have a cover  $X \xrightarrow{f} Y \in \mathrm{D}(M)\text{-Cov}(Y)$ . We want to define, for any  $m \in M$  a map  $h_{f,m} = h_{X/Y,m}: Y \rightarrow \{0, 1\}$ . Let  $q \in Y$  and denote by  $C$  the 'maximal torsor' of  $\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}$  (see 3.23). Also let  $p \in f^{-1}(q)$  and set  $p_C = p \cap C$ . We know that  $(\mathcal{O}_{X,q})_p = (\mathcal{O}_{X,q})_{p_C} = B$  and that  $B \in \mathrm{D}(M/\mathcal{H}_f(q))\text{-Cov}(C_{p_C})$  with  $H_{B/C_{p_C}} = 0$ . If we denote by  $\overline{m}$  the image of  $m$  in  $M/\mathcal{H}_f(q)$  we can define:

**Definition 3.27.** With notation above we set

$$h_{f,m}(q) = \begin{cases} 0 & \text{if } m \in \mathcal{H}_f(q) \\ \dim_{C_{p_C}/p_C}(m_B/m_B^2)_{\overline{m}} & \text{otherwise} \end{cases}$$

We also set

$$h_f(q) = \dim_{C_{p_C}/p_C}(m_B/m_B^2) - \dim_{C_{p_C}/p_C}(m_B/m_B^2)_0 = \left( \sum_{m \in M} h_{f,m}(q) \right) / |\mathcal{H}_f(q)|$$

If  $\mathcal{E} \in K_+^\vee$  we set  $h_{\mathcal{E},m} = h_{f,m}$ ,  $h_{\mathcal{E}} = h_f \in \mathbb{N}$  where  $f$  is the cover  $\mathrm{Spec} A \rightarrow \mathrm{Spec} k$  and  $A$  is the algebra given by multiplication  $0^\mathcal{E}$  over some field  $k$ .

**Lemma 3.28.** *Let  $(A, m_A)$  be a local ring,  $B \in \mathrm{D}(M)\text{-Cov}(A)$  given by multiplication  $\psi$  and  $t \in M$ . Then  $h_{B/A,t} = h_{B/A,t}(m_A)$  is well defined and  $h_{B/A,t} = 1$  if and only if  $t \notin H_{B/A}$  and for any  $u, n \in M - H_{B/A}$  such that  $u + n \equiv t \pmod{H_{A/B}}$  we have  $\psi_{u,n} \notin A^*$ .*

*Proof.* Let  $C$  be the maximal torsor of the extension  $B/A$  and  $p$  be a maximal prime of it. We use notation from 3.22. For any  $l \in M - H_{B/A}$  we have a surjective map

$$k(p) = (m_{B_p}/pC_p)_{\bar{l}} \longrightarrow (m_{B_p}/m_{B_p}^2)_{\bar{l}}$$

and so  $\dim_{k(p)}(m_{B_p}/m_{B_p}^2)_{\bar{l}} \in \{0, 1\}$ , where  $\bar{l}$  is the image of  $l$  through the projection  $M \longrightarrow M/H_{A/B}$ . If we prove the last part of the statement clearly we will also have that  $h_{B/A,t}$  is well defined. If  $t \in H_{B/A}$  then  $h_{B/A,t} = 0$ , while if there exist  $u, n$  as in the statement such that  $\psi_{u,n} \in A^*$ , then  $w_{\bar{t}} \in C_p^* w_{\bar{u}} w_{\bar{n}} \subseteq m_{B_p}^2$  and again  $h_{B/A,t} = 0$ . On the other hand if  $h_{B/A,t} = 0$  and  $t \notin H_{B/A}$  then  $w_{\bar{t}} \in m_{B_p}^2$  and therefore we have a writing

$$w_{\bar{t}} = bx + \sum_{\bar{u}, \bar{n} \neq 0} b_{\bar{u}, \bar{n}} w_{\bar{u}} w_{\bar{n}} \text{ with } b, b_{\bar{u}, \bar{n}} \in B_p, x \in m_{C_p}$$

The second sum splits as a sum of products of the form  $c_{s, \bar{u}, \bar{n}} w_s w_{\bar{u}} w_{\bar{n}}$  with  $s + \bar{u} + \bar{n} = \bar{t}$  and  $c_{s, \bar{u}, \bar{n}} \in C_p$ . Since  $C_p$  is local, one of these monomials generates  $C_p w_{\bar{t}}$ . In this case, if  $s + \bar{u} = 0$  then  $\bar{u} \in H_{B_p/C_p} = 0$  which is not the case. So we have a writing

$$w_{\bar{t}} = \lambda w_{\bar{u}} w_{\bar{n}} = \lambda \psi'_{\bar{u}, \bar{n}} w_{\bar{t}} \implies \psi'_{\bar{u}, \bar{n}} \in C_p^*$$

where  $\bar{u}, \bar{n} \neq 0$  and  $\bar{u} + \bar{n} = \bar{t}$ . Since  $\psi'_{\bar{u}, \bar{n}}$  and  $\psi_{u,n}$  differs by an element of  $C^*$  thanks to 3.22, it follows that  $\psi_{u,n} \in A^*$ .  $\square$

**Proposition 3.29.** *We have maps*

$$\begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{h_m} & \{0, 1\} \\ B/k \mid \longrightarrow & & h_{B/k, m} \end{array} \qquad \begin{array}{ccc} |\mathrm{D}(M)\text{-Cov}| & \xrightarrow{h} & \mathbb{N} \\ B/k \mid \longrightarrow & & h_{B/k} \end{array}$$

such that, if  $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$  is given by  $X \xrightarrow{f} Y$ , then  $h_{f, m} = h_m \circ u$  and  $h_f = h \circ g$ .

*Proof.* Taking into account 3.28 and 3.24, it's enough to note that if  $A$  is a local ring,  $B \in \mathrm{D}(M)\text{-Cov}(A)$  is given by multiplications  $\psi$  and  $\pi: A \longrightarrow A/m_A \longrightarrow k$  is a morphism, where  $k$  is a field, then  $\psi_{u,v} \in A^* \iff \pi(\psi_{u,v}) \neq 0$  and  $H_{B/A} = H_{B \otimes_A k/k}$ .  $\square$

**Corollary 3.30.** *Under the hypothesis of 3.26,  $\{m \in M \mid h_{B/A, m} = 1\}$  is the minimum subset of  $M$  such that  $B$  is generated as an  $A$ -algebra in those degrees. In particular  $B$  is generated in  $h_{B/A}$  degrees.*

**Proposition 3.31.** *Let  $(A, m_A)$  be a local ring,  $B \in \mathrm{D}(M)\text{-Cov}(A)$  and  $C$  the maximal torsor of  $B/A$ . Then*

$$h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$$

for any maximal prime  $p$  of  $B$ . In particular if  $(|H_{B/A}|, \mathrm{char} A/m_A) = 1$  we also have  $h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/A} \otimes_B k(p)$  for any maximal prime  $p$  of  $B$ .

*Proof.* If  $A$  is any ring and  $B \in \mathrm{D}(M)\text{-Cov}(A)$  is given by basis  $\{v_m\}_{m \in M}$  and multiplication  $\psi$  one sees from the universal property that

$$\Omega_{B/A} = B^M / \langle e_0, v_n e_m + v_m e_n - \psi_{m,n} e_{m+n} \rangle$$

Now consider  $B \in \mathrm{D}(M/H)\text{-Cov}(C)$ , where  $H = H_{B/A}$  and let  $p$  be a maximal prime of  $B$ . Following the notation of 3.22, we have that  $w_m \in p$  for any  $m \in M/H - \{0\}$  and  $\psi'_{m,n} \in p \iff \psi_{m,n} \in m_A$ . So  $\Omega_{B/C} \otimes_B k(p)$  is free on the  $e_m$  for  $m \in M/H - \{0\}$  such that for any  $u, n \in M/H - \{0\}$ ,  $u + n = m$  implies  $\psi_{u,n} \notin A^*$ , that are exactly  $h_{B/A}(m_A)$  thanks to 3.28.  $\square$

**Corollary 3.32.** *The function  $h$  is upper semicontinuous.*

*Proof.* Let  $X \xrightarrow{f} Y$  be an  $D(M)$ -cover and  $q \in Y$ . Set  $r = h_f(q)$  and  $H = \mathcal{H}_f(q)$ . We can assume that  $Y = \operatorname{Spec} A$ ,  $X = \operatorname{Spec} B$  with graded basis  $\{v_m\}_{m \in M}$  and multiplication  $\psi$  and that  $\psi_{m,-m} \in A^*$  for any  $m \in H$ . Set  $C = A[v_m]_{m \in H}$ .  $C_q$  is the maximal torsor of  $B_q/A_q$  and so, if  $p \in X$  is a point over  $q$ , we have  $r = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$ . Finally let  $U \subseteq X$  be an open neighborhood of  $p$  such that  $\dim_{k(p')} \Omega_{B/C} \otimes_B k(p') \leq r$  for any  $p' \in U$  and  $V = f(U)$ . We want to prove that  $h \leq r$  on  $V$ . Indeed given  $q' = f(p') \in V$ , if  $D$  is the maximal torsor of  $B_{q'}/A_{q'}$ , we have  $C_{q'} \subseteq D \subseteq B_{q'}$ . So

$$h_f(q') = \dim_{k(p')} \Omega_{B_{q'}/D} \otimes_{B_{q'}} k(p') \leq \dim_{k(p')} \Omega_{B_{q'}/C_{q'}} \otimes_{B_{q'}} k(p') \leq r$$

□

*Remark 3.33.* The 0 section  $R_M \rightarrow \mathbb{Z}$  induces a closed immersion

$$\operatorname{Pic}^{|M|-1} \simeq \operatorname{BT} = [\operatorname{Spec} \mathbb{Z}/\mathcal{T}] \subseteq [\operatorname{Spec} R_M/\mathcal{T}] \simeq D(M)\text{-Cov}$$

where  $\mathcal{T} = D(\mathbb{Z}^M / \langle e_0 \rangle)$ .

**Proposition 3.34.** *The following results hold:*

- (1)  $\{h = 0\} = |\operatorname{BD}(M)|$ ;
- (2)  $\{h \geq |M|\} = \emptyset$ ;
- (3)  $\{h = |M| - 1\} = |\operatorname{BD}(\mathbb{Z}^M / \langle e_0 \rangle)|$  (see 3.33)

*Proof.* If  $X \xrightarrow{f} Y$  is a  $D(M)$ -torsor, clearly  $h_f = 0$ . So 1) and 2) follow from 3.30. Finally, if  $B \in D(M)\text{-Cov}(k)$  with multiplication  $\psi$ ,  $h_{B/k} = |M| - 1$  if and only if  $H_{B/k} = 0$  and  $h_{B/k,m} = 1$  for any  $m \in M - \{0\}$ . This means that  $\psi_{m,n} = 0$  for any  $m, n \neq 0$  by 3.28. □

In particular, setting  $U_i = \{h \leq i\}$ , we obtain a stratification  $\operatorname{BD}(M) = U_0 \subseteq U_1 \subseteq \dots \subseteq U_{|M|-1} = D(M)\text{-Cov}$  of  $D(M)\text{-Cov}$  by open substacks.

**3.3. The locus  $h \leq 1$ .** In this subsection we want to describe  $D(M)$ -covers with  $h \leq 1$ . This means that 'up to torsors' we have a graded  $M$ -algebra generated over the base ring in one degree. We will see that  $\{h \leq 1\}$  is a smooth open substack of  $\mathcal{Z}_M$  determined by a special class of explicit smooth integral extremal rays of  $K_+$ . This will allow to give a description of covers over locally noetherian and locally factorial scheme  $X$  with  $(\operatorname{char} X, |M|) = 1$  whose total space is normal. This result, when  $X$  is a smooth algebraic variety over an algebraic closed field  $k$  with  $(\operatorname{char} k, |M|) = 1$ , was already proved in [Par91, Theorem 2.1, Corollary 3.1].

*Notation 3.35.* Given  $\mathcal{E} \in K_+^\vee$  we will write  $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$ . Since  $K \otimes \mathbb{Q} \simeq \mathbb{Q}^M / \langle e_0 \rangle$  we will also write  $\mathcal{E}_m = \mathcal{E}(e_m) \in \mathbb{Q}$ , so that  $\mathcal{E}_{m,n} = \mathcal{E}_m + \mathcal{E}_n - \mathcal{E}_{m+n}$ . Given a group homomorphism  $\eta: M \rightarrow N$  we will denote by  $\eta_*: K_M \rightarrow K_N$  the homomorphism such that  $\eta_*(v_{m,n}) = v_{\eta(m), \eta(n)}$  for all  $m, n \in M$ .

*Remark 3.36.* Let  $A$  be a ring and consider a sequence  $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$ . An element of  $\mathcal{F}_{\underline{\mathcal{E}}}(A)$  coming from the atlas (see 2.14) is given by a pair  $(\underline{z}, \lambda)$  where  $\underline{z} = z_1, \dots, z_r \in A$  and  $\lambda: K \rightarrow A^*$ . The image of this object under  $\pi_{\underline{\mathcal{E}}}$  is the algebra whose multiplication is given by  $\psi_{m,n} = \lambda_{m,n}^{-1} z_1^{\mathcal{E}_{m,n}^1} \dots z_r^{\mathcal{E}_{m,n}^r}$ .

**Lemma 3.37.** *Let  $\eta: M \rightarrow N$  be a surjective morphism and  $\underline{\mathcal{E}}$  be a sequence in  $(K_{+N})^\vee$ . Then  $\underline{\mathcal{E}}$  is a smooth sequence for  $N$  if and only if  $\underline{\mathcal{E}} \circ \eta_*$  is a smooth sequence for  $M$ .*

*Proof.* We want to apply 2.39. Therefore we have to prove that  $\eta_*(K_{+M}) = K_{+N}$ , which is clear, and that  $\operatorname{Ker} \eta_* = \langle \operatorname{Ker} \eta_* \cap K_{+N} \rangle$ . Consider the map  $f: \mathbb{Z}^M / \langle e_0 \rangle \rightarrow \mathbb{Z}^N / \langle e_0 \rangle$  given by  $f(e_m) = e_{\eta(m)}$  and set  $H = \operatorname{Ker} \eta$ . Clearly  $f|_{K_M} =$



$\eta_*$ . It is easy to check that  $G = \langle v_{m,n} \text{ for } m \in H \rangle_{\mathbb{Z}} \subseteq \text{Ker } \eta^* \subseteq \text{Ker } f$  and that  $\text{Ker } f / \text{Ker } \eta_* \simeq H$ . So in order to conclude, it is enough to note that the map  $H \rightarrow \text{Ker } f / G$  sending  $h$  to  $e_h$  is a surjective group homomorphism since we have relations  $e_h + e_{h'} - e_{h+h'} = v_{h,h'}$  and  $e_{m+h} - e_m = e_h - v_{m,h}$  for  $m \in M$  and  $h, h' \in H$ .  $\square$

**Proposition 3.38.** *Let  $\eta: M \rightarrow \mathbb{Z}/l\mathbb{Z}$  be a surjective homomorphism with  $l > 1$ . Then*

$$\mathcal{E}^\eta(v_{m,n}) = \begin{cases} 0 & \text{if } \eta(m) + \eta(n) < l \\ 1 & \text{otherwise} \end{cases}$$

*defines a smooth integral extremal ray for  $K_+$ .*

*Proof.*  $\mathcal{E}^\eta \in K_+^\vee$  because, if  $\sigma: \mathbb{Z}/l\mathbb{Z} \rightarrow \mathbb{N}$  is the obvious section,  $\mathcal{E}^\eta$  is the restriction of the map  $\mathbb{Z}^M / \langle e_0 \rangle \rightarrow \mathbb{Z}$  sending  $e_m$  to  $\sigma(\eta(m))$ . In order to conclude the proof, we will apply 3.37 and 2.38. Set  $N = \mathbb{Z}/l\mathbb{Z}$ . One clearly has  $\mathcal{E}^\eta = \mathcal{E}^{\text{id}} \circ \eta_*$  and so we can assume  $M = \mathbb{Z}/l\mathbb{Z}$  and  $\eta = \text{id}$ . In this case one can check that  $v_{1,1}, v_{1,2}, \dots, v_{1,l-1}$  is a  $\mathbb{Z}$ -base of  $K$  such that  $\mathcal{E}^\eta(v_{1,j}) = 0$  if  $j < l-1$ ,  $\mathcal{E}^\eta(v_{1,l-1}) = 1$ .  $\square$

Those particular rays have been already defined in [Par91, Equation 2.2].

*Notation 3.39.* If  $\phi: \tilde{K}_+ \rightarrow \mathbb{Z}^M / \langle e_0 \rangle$  is the usual map we set  $\mathcal{Z}_M^\mathcal{E} = \mathcal{X}_\phi^\mathcal{E}$  (see definition 2.18) for any sequence  $\mathcal{E}$  of elements of  $K_+^\vee$ . Remember that if  $\mathcal{E}$  is a smooth sequence then  $\mathcal{Z}_M^\mathcal{E}$  is a smooth open subset of  $\mathcal{Z}_M$  (see 2.41) and its points have the description given in 2.42.

Set  $\Phi_M$  for the union over all  $d > 1$  of the sets of surjective maps  $M \rightarrow \mathbb{Z}/d\mathbb{Z}$ .

**Theorem 3.40.** *Let  $\mathcal{E} = (\mathcal{E}^\eta)_{\eta \in \Phi_M}$ . We have*

$$\{h \leq 1\} = \bigcup_{\eta \in \Phi_M} \mathcal{Z}_M^{\mathcal{E}^\eta}$$

*In particular  $\{h \leq 1\} \subseteq \mathcal{Z}_M^{\text{sm}}$  and  $\pi_\mathcal{E}$  induces an equivalence of categories*

$$\{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_\mathcal{E} \mid V(z_\eta) \cap V(z_\mu) = \emptyset \text{ if } \eta \neq \mu\} = \pi_\mathcal{E}^{-1}(\{h \leq 1\}) \xrightarrow{\sim} \{h \leq 1\}$$

*Proof.* The last part of the statement follows from the first one just applying 2.45 with  $\Theta = \{(\mathcal{E}^\eta)\}_{\eta \in \Phi_M}$ . Let  $k$  be an algebraically closed field and  $B \in \text{D}(M)\text{-Cov}(k)$  with graded basis  $\{v_m\}_{m \in M}$  and multiplication  $\psi$ .

$\supseteq$ . Assume  $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}(k)$ . If  $B$  is a torsor we will have  $h_{B/k} = 0$ . Otherwise we can write  $\psi = \xi 0^{\mathcal{E}^\eta}$  for some  $\xi: K \rightarrow k^*$ . Up to change  $\text{Spec } k$  with a geometrical point of the maximal torsor of  $B/k$ , we can assume that  $M = \mathbb{Z}/d\mathbb{Z}$  and  $\eta = \text{id}$ . In particular  $h_{B/k} = 0$  and, from the definition of  $\mathcal{E}^{\text{id}}$ , we get  $B \simeq k[x]/(x^d)$ . So  $h_{B/k} = \dim_k m_B / m_B^2 = 1$ .

$\subseteq$ . Assume  $h_{B/k} = 1$ . Set  $C$  for the maximal torsor of  $B/k$  (see 3.23),  $H = H_{B/k}$  and  $l = |M/H|$ .  $h_{B/k} = 1$  means that there exists a unique  $\bar{r} \in M/H$  (where  $r \in M$ ) such that  $h_{B/k,r} = 1$  and so  $C_q[v_r] = B_q \simeq C_q[x]/(x^l)$  for all (maximal) primes  $q$  of  $C$ . In particular  $B = C[v_r] \simeq C[x]/(x^l)$  and  $\bar{r}$  generates  $M/H$ . Let  $\eta: M \rightarrow M/H \simeq \mathbb{Z}/l\mathbb{Z}$  be the projection. We want to prove that  $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}$ . Up to change  $k$  with a geometrical point of some fppf extension of  $k$ , we can assume  $C = k[H]$ , i.e.  $v_h v_{h'} = v_{h+h'}$  if  $h, h' \in H$ . Finally the elements  $v_h v_r^i$  for  $h \in H$  and  $0 \leq i < l$  define an  $M$ -graded basis of  $B/k$  whose associated multiplication is  $0^{\mathcal{E}^\eta}$ .  $\square$

**Theorem 3.41.** *Let  $\mathcal{E} = (\mathcal{E}^\eta)_{\eta \in \Phi_M}$  and let  $X$  be a locally noetherian and locally factorial scheme. Set  $\mathcal{C}_X^1 = \{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_\mathcal{E}(X) \mid \text{codim}_X V(z_\eta) \cap V(z_\mu) \geq 2 \text{ if } \eta \neq \mu\}$  and  $\mathcal{D}_X^1 = \{Y \xrightarrow{f} X \in \text{D}(M)\text{-Cov}(X) \mid h_f(p) \leq 1 \forall p \in X \text{ with } \text{codim}_p X \leq 1\}$ .*



Then  $\pi_{\underline{\mathcal{E}}}$  induces an equivalence of categories

$$\mathcal{D}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{C}_X^1) \xrightarrow{\sim} \mathcal{C}_X^1$$

*Proof.* Apply 2.53 with  $\Theta = \{(\mathcal{E}^\eta)\}_{\eta \in \Phi_M}$ .  $\square$

**Theorem 3.42.** Let  $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta \in \Phi_M}$  and let  $X$  be a locally noetherian and locally factorial scheme without isolated points and  $(\text{char } X, |M|) = 1$ , i.e.  $1/|M| \in \mathcal{O}_X(X)$ . Set

$$\text{Reg}_X^1 = \{Y/X \in \text{D}(M)\text{-Cov}(X) \mid Y \text{ regular in codimension 1}\}$$

and

$$\widetilde{\text{Reg}}_X^1 = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \begin{array}{l} \forall \mathcal{E} \neq \delta \in \underline{\mathcal{E}} \text{ codim}_X V(z_{\mathcal{E}}) \cap V(z_{\delta}) \geq 2 \\ \forall \mathcal{E} \in \underline{\mathcal{E}} \forall p \in X^{(1)} v_p(z_{\mathcal{E}}) \leq 1 \end{array} \right\}$$

Then we have an equivalence of categories

$$\widetilde{\text{Reg}}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\text{Reg}_X^1) \xrightarrow{\sim} \text{Reg}_X^1$$

*Proof.* We will make use of 3.41. If  $Y \xrightarrow{f} X \in \text{Reg}_X^1$ ,  $p \in Y^{(1)}$  and  $q = f(p)$  then  $h_f(q) \leq \dim_{k(p)} m_p/m_p^2 = 1$ . So  $\text{Reg}_X^1 \subseteq \mathcal{D}_X^1$ . So we have only to check that  $\widetilde{\text{Reg}}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\text{Reg}_X^1) \subseteq \mathcal{C}_X^1$ . In particular we can assume that  $X = \text{Spec } R$ , where  $R$  is a DVR. Let  $\chi \in \mathcal{C}_X^1$ ,  $A/R \in \mathcal{D}_X^1$  the associated covers,  $H = H_{A/R}$  and  $C$  be the maximal torsor of  $A/R$ . We have to prove that  $\chi \in \widetilde{\text{Reg}}_X^1$  if and only if  $A$  is regular in codimension 1. Since  $D_R(H)$  is etale over  $R$  so is also  $\text{Spec } C$ . It is so easy to check that, up to change  $R$  with a localization of  $C$  and  $M$  with  $M/H$ , we can assume that  $H = 0$ . Since  $\chi \in \mathcal{C}_X^1$ , the multiplication of  $A$  over  $R$  is of the form  $\psi = \mu z^{r_{\mathcal{E}\phi}}$ , where  $\mu: K \rightarrow R^*$  is an  $M$ -torsor,  $z$  is a parameter of  $A$ ,  $\phi: M \rightarrow \mathbb{Z}/l\mathbb{Z}$  is an isomorphism and  $r = v_R(z_{\mathcal{E}\phi})$ . Moreover  $v_R(z_{\mathcal{E}\psi}) = 0$  if  $\psi \neq \phi$ . Up to change  $M$  with  $\mathbb{Z}/l\mathbb{Z}$  through  $\phi$  we can assume  $\phi = \text{id}$ . Finally, since  $\mu$  induces an (fppf) torsor which is etale over  $R$ , up to change  $R$  with an etale neighborhood, we can assume  $\mu = 1$ . After these reductions we have  $A = R[X]/(X^{|M|} - z^r)$  which is regular in codimension 1 if and only if  $r = 1$ .  $\square$

*Remark 3.43.* In the theorem above one can replace the condition 'regular in codimension 1' in the definition of  $\text{Reg}_X^1$  with 'normal' thanks to Serre's conditions, since all the fibers involved are Gorenstein.

*Remark 3.44.* Theorem 3.42 is a rewriting of Theorem 2.1 and Corollary 3.1 of [Par91] extended to locally noetherian and locally factorial schemes without isolated points, where an object of  $\mathcal{F}_{\underline{\mathcal{E}}}(X)$  is called a building data.

#### 4. THE LOCUS $h \leq 2$

In this section we want to give a characterization of the open subset  $\{h \leq 2\} \subseteq \text{D}(M)\text{-Cov}$  as done in 3.41 for  $\{h \leq 1\}$ . The general problem we want to solve can be stated as follows.

**Problem 4.1.** Find a sequence of smooth integral extremal rays  $\underline{\mathcal{E}}$  for  $M$  and a collection  $\Theta$  of smooth sequences with rays in  $\underline{\mathcal{E}}$  such that (see 3.39)

$$\{h \leq 2\} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{Z}_M^{\underline{\delta}}$$

or, equivalently, such that, for any algebraically closed field  $k$ , the algebras  $A \in \text{D}(M)\text{-Cov}(k)$  with  $h_{A/k} \leq 2$  are exactly the algebras associated to a multiplication of the form  $\psi = \omega 0^{\mathcal{E}}$  where  $\omega: K_+ \rightarrow k^*$  and  $\mathcal{E} \in \prec \underline{\delta} \succ_{\mathbb{N}}$  for some  $\underline{\delta} \in \Theta$ .

For example in the case  $h \leq 1$  the analogous problem is solved taking  $\underline{\mathcal{E}} = (\mathcal{E}^\phi)_{\phi \in \Phi_M}$  and  $\Theta = \{(\mathcal{E}) \text{ for } \mathcal{E} \in \underline{\mathcal{E}}\}$  (see 3.40). Once we have found a pair  $\underline{\mathcal{E}}, \Theta$  as in 4.1 we can formally apply theorems 2.45 and 2.53. This is done in theorems 4.41 and 4.44.

The first problem (see 4.8) to solve is to describe  $M$ -graded algebras over a field  $k$  (yielding  $D(M)$ -covers) generated in two degrees. As for the case  $h \leq 1$ , where we can reduce to study  $\mathbb{Z}/d\mathbb{Z}$ -graded algebras generated in degree 1 and then considering surjective maps  $M \rightarrow \mathbb{Z}/d\mathbb{Z}$ , also in the case  $h \leq 2$  we can restrict our attention to the study of algebras generated in two degrees  $m, n \in M$ . It is easy to see (4.6) that a group with two marked elements generating it is canonically isomorphic to

$$M_{r,\alpha,N} = \mathbb{Z}^2 / (r, -\alpha), (0, N), e_1, e_2$$

for some natural numbers  $0 \leq \alpha < N$ ,  $r > 0$ .

So our problem become the study of  $D(M_{r,\alpha,N})$ -covers  $A$  over a field  $k$  generated in degrees  $e_1, e_2$  and such that  $H_{A/k} = 0$  (thanks to 3.30, when  $H_{A/k} = 0$  the number  $h_{A/k}$  is simply the minimum number of degrees in which  $A$  is generated). These algebras are related to the following set: given  $q \in \mathbb{Z}$ , define  $d_q$  as the only integer  $0 < d_q \leq N$  such that  $d_q \equiv -q\alpha \pmod{N}$  and define

$$\Omega_{N-\alpha,N} = \{0 < q \leq o(\alpha, \mathbb{Z}/N\mathbb{Z}) = N/(N, \alpha) \mid d_{q'} < d_q \text{ for any } 0 < q' < q\}$$

The point is that if we take  $z_A$  as the minimum  $h > 0$  for which there exist  $s \in \mathbb{N}$  and  $\lambda \in k$  such that  $hm = sn$  and  $v_m^h = \lambda v_n^s$ , then  $\bar{q}_A = z_A/r \in \Omega_{N-\alpha,N}$  (see 4.22). We will show that also the converse holds, in the sense that for any  $\bar{q} \in \Omega_{N-\alpha,N}$  there exists a  $D(M_{r,\alpha,N})$ -cover  $A/k$  as above with  $\bar{q}_A = \bar{q}$ . The precise statement is 4.30. This gives a solution to the first problem and also suggests how to proceed for the next one, i.e. find the sequence  $\underline{\mathcal{E}}$  of problem 4.1.

It turns out that the integral extremal rays  $\mathcal{E}$  for  $M$  such that  $h_{\mathcal{E}} = 2$  correspond to particular sequences of the form  $\chi = (r, \alpha, N, \bar{q}, \phi)$ , where  $0 \leq \alpha < N$ ,  $r > 0$ ,  $\bar{q} \in \Omega_{N-\alpha,N}$  and  $\phi: M \rightarrow M_{r,\alpha,N}$  is a surjective map (see 4.39). The sequence of smooth integral extremal rays “needed” to describe the substack  $\{h \leq 2\}$  is composed by the “old” rays  $(\mathcal{E}^\eta)_{\eta \in \Phi_M}$  and by these new rays. Finally the smooth sequences in the family  $\Theta$  of problem 4.1 will be all given by elements of the dual basis of particular  $\mathbb{Z}$ -basis of  $K$  (see 4.33).

In the last subsection we will see (Theorem 4.54) that the  $D(M)$ -covers of a locally noetherian and locally factorial scheme with no isolated points and with  $(\text{char } X, |M|) = 1$  whose total space is normal crossing in codimension 1 can be described in the spirit of classification 3.42 and extending this result.

**4.1. Good sequences.** In this subsection we provide some general technical results in order to work with  $M$ -graded algebras over local rings. So we will consider given a local ring  $D$ , a sequence  $\underline{m} = m_1, \dots, m_r \in M$  and  $C \in D(M)\text{-Cov}(D)$  generated in degrees  $m_1, \dots, m_r$ . Since  $\text{Pic Spec } D = 0$  for any  $u \in M$  we have  $C_u \simeq D$ . Given  $u \in M$ , we will call  $v_u$  a generator of  $C_u$  and we will also use the abbreviation  $v_i = v_{m_i}$ . Moreover, if  $\underline{A} = (A_1, \dots, A_r) \in \mathbb{N}^r$  we will also write

$$v^{\underline{A}} = v_1^{A_1} \dots v_r^{A_r}$$

**Definition 4.2.** A sequence for  $u \in M$  is a sequence  $\underline{A} \in \mathbb{N}^r$  such that  $A_1 m_1 + \dots + A_r m_r = u$ . Such a sequence will be called *good* if the map  $C_{m_1}^{A_1} \otimes \dots \otimes C_{m_r}^{A_r} \rightarrow C_u$  is surjective, i.e.  $v^{\underline{A}}$  generates  $C_u$ . If  $r = 2$  we will talk about pairs instead of sequences.

*Remark 4.3.* Any  $u \in M$  admits a good sequence since, otherwise, we will have  $C_u = (D[v_1, \dots, v_r])_u \subseteq m_D C_u$ . If  $\underline{A}$  is a good sequence and  $\underline{B} \leq \underline{A}$ , then also  $\underline{B}$  is a good.

**Lemma 4.4.** *Let  $\underline{A}, \underline{B}$  be two sequences for some element of  $M$  and assume that  $\underline{A}$  is good. Set  $\underline{E} = \min(\underline{A}, \underline{B}) = (\min(A_1, B_1), \dots, \min(A_r, B_r))$  and take  $\lambda \in D$ . Then*

$$v^{\underline{B}} = \lambda v^{\underline{A}} \implies v^{\underline{B}-\underline{E}} = \lambda v^{\underline{A}-\underline{E}}$$

*Proof.* Clearly we have  $v^{\underline{E}}(v^{\underline{B}-\underline{E}} - \lambda v^{\underline{A}-\underline{E}}) = 0$ . On the other hand, since  $\underline{A} - \underline{E}$  is a good sequence, there exists  $\mu \in D$  such that  $v^{\underline{B}-\underline{E}} = \mu v^{\underline{A}-\underline{E}}$ . Since  $\underline{A}$  is a good sequence, substituting we get  $v^{\underline{A}}(\mu - \lambda) = 0 \implies \mu = \lambda$ .  $\square$

#### 4.2. $M$ -graded algebras generated in two degrees.

**Definition 4.5.** Given  $0 \leq \alpha < N$ ,  $r > 0$  set  $M_{r,\alpha,N} = \mathbb{Z}^2 / (r, -\alpha), (0, N)$ .

**Proposition 4.6.** *A finite abelian group  $M$  with two marked elements  $m, n \in M$  generating it is canonically isomorphic to  $(M_{r,\alpha,N}, e_1, e_2)$  where  $r = \min\{s > 0 \mid sm \in \langle n \rangle\}$ ,  $rm = \alpha n$  and  $N = o(n)$ . Moreover we have:  $|M| = Nr$ ,  $o(m) = rN/(\alpha, N)$  and*

$$m, n \neq 0 \text{ and } m \neq n \iff N > 1 \text{ and } (r > 1 \text{ or } \alpha > 1)$$

*Proof.* We have

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} r & 0 \\ -\alpha & N \end{pmatrix}} \mathbb{Z}^2 \longrightarrow M_{r,\alpha,N} \longrightarrow 0 \implies |M_{r,\alpha,N}| = \left| \det \begin{pmatrix} r & 0 \\ -\alpha & N \end{pmatrix} \right| = rN$$

and clearly  $e_1, e_2$  generate  $M$ . Moreover  $M_{r,\alpha,N} / \langle e_2 \rangle \simeq \mathbb{Z}/r\mathbb{Z}$  and therefore  $r$  is the minimum such that  $re_1 \in \langle e_2 \rangle$ . Finally it is easy to check that  $N = o(e_2)$ . If now  $M, r, \alpha, N$  are as in the statement, there exists a unique map  $M_{r,\alpha,N} \longrightarrow M$  sending  $e_1, e_2$  to  $m, n$ . This map is an isomorphism since it is clearly surjective and  $|M| = o(m)o(n)/|\langle m \rangle \cap \langle n \rangle| = o(n)r = |M_{r,\alpha,N}|$ . The last equivalence in the statement is now easy to prove.  $\square$

*Notation 4.7.* In this subsection we will fix a finite abelian group  $M$  generated by two elements  $0 \neq m, n \in M$  such that  $m \neq n$ . Up to isomorphism, this means  $M = M_{r,\alpha,N}$  with  $m = e_1$ ,  $n = e_2$  and conditions  $0 \leq \alpha < N$ ,  $r > 0$ ,  $N > 1$ , ( $r > 1$  or  $\alpha > 1$ ).

We will write  $d_q$  for the only integers  $0 < d_q \leq N$  such that  $qrm + d_q n = 0$ , for  $q \in \mathbb{Z}$ , or, equivalently,  $d_q \equiv -q\alpha \pmod{N}$ .

**Problem 4.8.** Let  $k$  be a field. We want to describe, up to isomorphism, algebras  $A \in \mathcal{D}(M)\text{-Cov}(k)$  such that  $A$  is generated in degrees  $m, n$  and  $H_{A/k} = 0$ . Thanks to 3.30, this is equivalent to asking for an algebra  $A$  such that  $H_{A/k} = 0$  and

$$\{l \in M \mid h_{A/k,l} = 1\} \subseteq \{m, n\}$$

The solution of this problem is contained in 4.30.

In this subsection we will fix an algebra  $A$  as in 4.8.  $\{v_l\}_{l \in M}$  will be a graded basis of  $A$  and  $\psi$  the associated multiplication. Note that  $H_{A/k} = 0$  means  $v_m, v_n \notin A^*$ .

**Definition 4.9.** Define

$$z = \min\{h > 0 \mid v_m^h = \lambda v_n^i \text{ for some } \lambda \in k \text{ and } hm = in\}$$

$$x = \min\{h > 0 \mid v_n^h = \mu v_m^i \text{ for some } \mu \in k \text{ and } hn = im\}$$

Denote by  $0 \leq y < o(n)$ ,  $0 \leq w < o(m)$  the elements such that  $zm = yn$ ,  $xn = wm$ , by  $\lambda, \mu \in k$  the elements such that  $v_m^z = \lambda v_n^y$ ,  $v_n^x = \mu v_m^w$ , with the convention that  $\lambda = 0$  if  $v_n^y = 0$  and  $\mu = 0$  if  $v_m^w = 0$ . Finally set  $\bar{q} = z/r$  and define the map of sets

$$\begin{aligned} \{0, 1, \dots, z-1\} &\xrightarrow{f} \{0, 1, \dots, o(n)\} \\ c &\longmapsto \min\{d \in \mathbb{N} \mid v_m^c v_n^d = 0\} \end{aligned}$$

We will also write  $\bar{q}_A, z_A, x_A, y_A, w_A, \lambda_A, \mu_A, f_A$  if necessary.

We will see that  $A$  is uniquely determined by  $\bar{q}$  and  $\lambda$  up to isomorphism.

**Lemma 4.10.** *Given  $l \in M$  there exists a unique good pair  $(a, b)$  for  $l$  with  $0 \leq a < z$ . Moreover  $0 \leq b < f(a)$ .*

*Proof. Existence.* We know that there exists a good pair  $(a, b)$  for  $l$  and we can assume that  $a$  is minimum. If  $a \geq z$  we can write  $v_m^a v_n^b = \lambda v_m^{a-z} v_n^{b+y}$ . Therefore  $\lambda \neq 0$  and  $(a-z, b+y)$  is a good pair for  $l$ , contradicting the minimality of  $a$ . Finally  $v_m^a v_n^b \neq 0$  means  $b < f(a)$ .

*Uniqueness.* Let  $(a, b), (a', b')$  be two good pairs for  $l$  and assume  $0 \leq a < a' < z$ . So there exists  $\omega \in k^*$  such that

$$v_m^a v_n^b = \omega v_m^{a'} v_n^{b'} \implies v_n^b = \omega v_m^{a'-a} v_n^{b'}$$

If  $b \geq b'$  then  $a' - a \geq z$  by definition of  $z$ , while if  $b < b'$  then  $v_n$  is invertible.  $\square$

**Definition 4.11.** Given  $l \in M$  we will write the associated good pair as  $(\mathcal{E}_l, \delta_l)$  with  $\mathcal{E}_l < z$ .  $\mathcal{E}, \delta$  will be considered as maps  $\mathbb{Z}^M / \langle e_0 \rangle \rightarrow \mathbb{Z}$  and, if necessary, we will also write  $\mathcal{E}^A, \delta^A$ .

*Notation 4.12.* Up to isomorphism, we can change the given basis to

$$v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$$

so that the multiplication  $\psi$  is given by

$$(4.1) \quad v_a v_b = v_m^{\mathcal{E}_a + \mathcal{E}_b} v_n^{\delta_a + \delta_b} = \psi_{a,b} v_m^{\mathcal{E}_{a+b}} v_n^{\delta_{a+b}} = \psi_{a,b} v_{a+b}$$

**Corollary 4.13.**  *$f$  is a decreasing function and*

$$(4.2) \quad f(0) + \dots + f(z-1) = |M|$$

*Proof.* If  $(a, b)$  is a pair such that  $0 \leq a < z$  and  $0 \leq b < f(a)$  then  $v_m^a v_n^b \neq 0$ , i.e.  $(a, b)$  is a good pair for  $am + bn$ . So

$$\sum_{c=0}^{z-1} f(c) = |\{(a, b) \mid 0 \leq a < z, 0 \leq b < f(a)\}| = |M|$$

$\square$

*Remark 4.14.* The following pairs are good:

$$(z-1)m : (z-1, 0), (x-1)n : (0, x-1), zm = yn : (0, y), xn = wm : (w, 0)$$

i.e.  $v_m^{z-1}, v_n^{x-1}, v_n^y, v_m^w \neq 0$ . In particular  $f(0) \geq x, y+1$  and  $f(c) > 0$  for any  $c$ . Indeed

$$\begin{aligned} v_m^{z-1} = \omega v_m^a v_n^b &\implies v_m^{z-1-a} = \omega v_n^b \implies a = z-1, b = 0 \\ v_m^z = \omega v_m^a v_n^b &\implies v_m^{z-a} = \omega v_n^b \implies a = 0, b = y \end{aligned}$$

where  $(a, b)$  are good pairs for the given elements and, by symmetry, we get the result.

*Remark 4.15.* If  $\lambda \neq 0$  or  $\mu \neq 0$  then  $x = y, z = w$  and  $\lambda\mu = 1$ . Assume for example  $\lambda \neq 0$ . If  $y = 0$  then  $v_m^z = \lambda \neq 0$  and so  $v_m$  is invertible. So  $y > 0$  and, since  $v_n^y = \lambda^{-1} v_m^z$ , we also have  $y \geq x$ . Now

$$0 \neq v_m^z = \lambda v_n^y = \lambda \mu v_n^{y-x} v_m^w$$

So  $\mu \neq 0$  and  $(y-x, w)$  is a good pair. As before  $w \geq z$  and therefore

$$\lambda \mu v_n^{y-x} v_m^{w-z} = 1 \implies y = x, w = z \text{ and } \lambda\mu = 1$$

**Lemma 4.16.** *Let  $a, b \in M$ . We have:*

- Assume  $\mathcal{E}_{a,b} > 0$ . If  $\delta_{a,b} \leq 0$  then  $\mathcal{E}_{a,b} \geq z$ ,  $\delta_{a,b} \geq -y$ . Moreover  $\psi_{a,b} \neq 0 \iff \lambda \neq 0, \mathcal{E}_{a,b} = z, \delta_{a,b} = -y (= -x)$  and in this case  $\psi_{a,b} = \lambda$ .
- Assume  $\mathcal{E}_{a,b} < 0$ . Then  $\mathcal{E}_{a,b} \geq -w, \delta_{a,b} \geq x$ . Moreover  $\psi_{a,b} \neq 0 \iff \mu \neq 0, \mathcal{E}_{a,b} = -w (= -z), \delta_{a,b} = x$  and in this case  $\psi_{a,b} = \mu$ .
- Assume  $\mathcal{E}_{a,b} = 0$ . Then we have  $\delta_{a,b} = 0$  and  $\psi_{a,b} = 1$  or  $\delta_{a,b} \geq o(n)$  and  $\psi_{a,b} = 0$ .

*Proof.* Set  $\psi = \psi_{a,b}$ . We start with the case  $\mathcal{E}_{a,b} > 0$ . From 4.1 we get

$$v_m^{\mathcal{E}_{a,b}} v_n^{\delta_{a,b}} = \psi v_n^{\delta_{a,b}}$$

If  $\delta_{a,b} > 0$  then  $v_m^{\mathcal{E}_{a,b}} v_n^{\delta_{a,b}} = \psi$  and so  $\psi = 0$  since  $v_m \notin A^*$ . If  $\delta_{a,b} \leq 0$  we instead have  $v_m^{\mathcal{E}_{a,b}} = \psi v_n^{-\delta_{a,b}}$  and so  $\mathcal{E}_{a,b} \geq z$ . If  $-\delta_{a,b} < y$  then  $(0, -\delta_{a,b})$  is good. So we can write

$$v_m^{\mathcal{E}_{a,b}-z} \lambda v_n^{y+\delta_{a,b}} = \psi \implies \psi = 0$$

since  $v_n$  is not invertible. If  $\delta_{a,b} \leq -y$  we have

$$0 \leq \mathcal{E}_{a,b} - z < z, 0 \leq -\delta_{a,b} - y < f(0), (\mathcal{E}_{a,b} - z)m = (-\delta_{a,b} - y)n, v_m^{\mathcal{E}_{a,b}-z} \lambda = \psi v_n^{-\delta_{a,b}-y}$$

and so both  $(\mathcal{E}_{a,b} - z, 0)$  and  $(0, -\delta_{a,b} - y)$  are good pair for the same element of  $M$ . Therefore we must have  $\mathcal{E}_{a,b} = z, \delta_{a,b} = -y$  and  $\psi = \lambda$ .

Now assume  $\mathcal{E}_{a,b} = 0$ . If  $\delta_{a,b} < 0$  then  $v_n^{-\delta_{a,b}} \psi = 1$  which is impossible. So  $\delta_{a,b} \geq 0$ . If  $\delta_{a,b} = 0$  clearly  $\psi = 1$ . If  $\delta_{a,b} > 0$  then  $v_n^{\delta_{a,b}} = \psi$  and so  $\psi = 0$  and  $\delta_{a,b} \geq o(n)$ .

Finally assume  $\mathcal{E}_{a,b} < 0$ . From 4.1 we get

$$v_n^{\delta_{a,b}} = \psi v_m^{-\mathcal{E}_{a,b}} v_n^{\delta_{a,b}}$$

We must have  $\delta_{a,b} > 0$  since  $v_m$  is not invertible. So  $v_n^{\delta_{a,b}} = \psi v_m^{-\mathcal{E}_{a,b}}$  and  $\delta_{a,b} \geq x$ , from which

$$v_n^{\delta_{a,b}-x} \mu v_m^w = \psi v_m^{-\mathcal{E}_{a,b}}$$

Note that, since  $0 \leq -\mathcal{E}_{a,b} \leq \mathcal{E}_{a,b} < z$ ,  $(-\mathcal{E}_{a,b}, 0)$  is a good pair. If  $w > -\mathcal{E}_{a,b}$  then  $\psi = 0$ . So assume  $w \leq -\mathcal{E}_{a,b}$ . Arguing as above we must have  $\delta_{a,b} = x, \mathcal{E}_{a,b} = -w$  and  $\psi = \mu$ .  $\square$

**Lemma 4.17.** *Define*

$$A' = k[s, t] / (s^z, s^c t^{f(c)} \text{ for } 0 \leq c < z)$$

*Then  $A' \in \mathcal{D}(M)\text{-Cov}(k)$  with graduation  $\deg s = m, \deg t = n$  and it satisfies the requests of 4.8, i.e.  $A'$  is generated in degrees  $m, n$  and  $H_{A'/k} = 0$ . Moreover we have*

$$\bar{q}_{A'} = \bar{q}_A, z_{A'} = z_A, y_{A'} = y_A, \mathcal{E}^{A'} = \mathcal{E}^A, \delta^{A'} = \delta^A, \lambda_{A'} = \mu_{A'} = 0, f_{A'} = f_A$$

*Proof.* Clearly the elements  $s^c t^d$  for  $0 \leq c < z, 0 \leq d < f(c)$  generates  $A'$  as a  $k$ -space. Since they are  $\sum_{c=0}^{z-1} f(c) = |M|$  and they all have different degrees, it's enough to prove that any of them are non-zero. So let  $(c', d')$  a pair as always. It is enough to show that  $B = k[s, t] / (s^{c'+1}, t^{d'+1}) \longrightarrow A' / (s^{c'+1}, t^{d'+1})$  is an isomorphism. But  $c' < z$  implies that  $s^z = 0$  in  $B$ . If  $c' < c$  then  $s^c t^{f(c)} = 0$  in  $B$  and finally if  $c' \geq c$  then  $d' + 1 \leq f(c') \leq f(c)$  and so  $s^c t^{f(c)} = 0$  in  $B$ .

$A'$  is clearly generated in degrees  $m, n$  and  $H_{A'/k} = 0$  since  $s^z = t^{f(0)} = 0$  and  $z, f(0) > 0$ .  $s^z = 0t^y$  implies that  $z' = z_{A'} \leq z$ . Assume by contradiction  $z' < z$ . From  $0 \neq s^{z'} = \lambda' t^{y'}$  we know that  $t^{y'} \neq 0$  so that  $y' < f(0)$ . Therefore  $(\mathcal{E}_{z'm}, \delta_{z'm}) = (z', 0) = (0, y')$  and so  $z' = 0$ , which is a contradiction. Then  $z' = z, y_{A'} = y' = y$ . Also  $s^z = 0t^y$  and  $t^y \neq 0$  imply  $\lambda_{A'} = 0$  and, thanks to 4.15,  $\mu_{A'} = 0$ . Finally by construction we also have  $\mathcal{E}^{A'} = \mathcal{E}, \delta^{A'} = \delta$  and  $f_{A'} = f$ .  $\square$

**Lemma 4.18.** *We have*

$$d_{\bar{q}} = \max_{1 \leq q \leq \bar{q}} d_q$$

*Proof.* Thanks to 4.17 we can assume  $\lambda = 0$  and, therefore,  $\mu = 0$ . So  $v_n^x = 0$ ,  $v_n^{x-1} \neq 0$  and  $v_n^y \neq 0$  imply  $y < x = f(0)$ . Let  $1 \leq q < \bar{q}$  and  $l = qr$ . We have  $(\mathcal{E}_l, \delta_l) = (qr, 0)$ . If  $N - d_q < x = f(0)$  then we will also have  $(\mathcal{E}_l, \delta_l) = (0, N - d_q)$  and so  $q = 0$ , which is not the case. So  $N - d_q \geq x > y = N - d_{\bar{q}} \implies d_q < d_{\bar{q}}$ .  $\square$

**Lemma 4.19.** *Define  $\hat{q}$  as the only integers  $0 \leq \hat{q} < \bar{q}$  such that*

$$d_{\hat{q}} = \min_{0 \leq q < \bar{q}} d_q$$

*If  $\lambda = 0$  we have  $d_{\hat{q}} \leq x = f(0)$  and  $f(c) = \begin{cases} x & \text{if } 0 \leq c < \hat{q}r \\ d_{\hat{q}} & \text{if } \hat{q}r \leq c < z \end{cases}$*

*Proof.* We want first prove that  $f(c) = \min(x, d_q \text{ for } 0 \leq qr \leq c)$ . Clearly we have the inequality  $\leq$  since  $v_n^x = v_m^{qr} v_n^{d_q} = 0$ . Set  $d = f(c)$  and let  $(a, b)$  a good pair for  $cm + dn$ , so that  $v_m^c v_n^d = 0 v_m^a v_n^b$ . We cannot have  $b \geq d$  since otherwise  $v_m^c = 0$  implies  $c \geq z$ . If  $a \geq c$  then  $v_n^d = 0$  and so  $d = f(c) \geq x$ . Conversely if  $a < c$  then  $0 \leq c - a = qr \leq c < z$  and  $0 < d - b = d_q \leq d = f(c)$ .

We are now ready to prove the writing of  $f$ . Note that the pairs  $(qr, d_q - 1)$ , with  $0 \leq q < \bar{q}$ , are all the possible pairs for  $-n$ . So there exists a unique  $0 \leq \tilde{q} < \bar{q}$  such that  $(\tilde{q}r, d_{\tilde{q}} - 1)$  is good. In particular if  $0 \leq q \neq \tilde{q} < \bar{q}$  we have a writing

$$v_m^{qr} v_n^{d_q-1} = 0 v_m^{\tilde{q}r} v_n^{d_{\tilde{q}}-1} \implies \begin{cases} q < \tilde{q} & \implies v_n^{d_q-1} = 0 & \implies d_q \geq x \\ q > \tilde{q} & \implies d_q > d_{\tilde{q}} \end{cases}$$

Since  $v_n^{d_{\tilde{q}}-1} \neq 0$  we must have  $d_{\tilde{q}} \leq x$ . This shows that  $\tilde{q} = \hat{q}$  and the writing of  $f$ . Finally If  $\bar{q} > 1$  then  $\hat{q} > 0$  and so  $d_{\hat{q}} \leq x = f(0)$  since  $f$  is a decreasing function. If  $\bar{q} = 1$  then  $\hat{q} = 0$  and so  $N = d_{\hat{q}} = f(0) \leq x \leq N$ .  $\square$

**Definition 4.20.** We will continue to use notation from 4.19 for  $\hat{q}$  and we will also write  $\hat{q}_A$  if necessary.

### 4.3. The invariant $\bar{q}$ .

**Lemma 4.21.** *Let  $\beta, N \in \mathbb{N}$ , with  $N > 1$ , and define  $d_q^\beta = d_q$ , for  $q \in \mathbb{Z}$ , the only integer  $0 < d_q \leq N$  such that  $d_q \equiv q\beta \pmod{N}$ . Set*

$$\Omega_{\beta, N} = \{0 < q \leq o(\beta, \mathbb{Z}/N\mathbb{Z}) = N/(N, \beta) \mid d_{q'} < d_q \text{ for any } 0 < q' < q\},$$

*set  $q_n$  for the  $n$ -th element of it and denote by  $0 \leq \hat{q} < q_n$  the only number such that*

$$d_{\hat{q}} = \min_{0 \leq q < q_n} d_q$$

*Then we have relations  $\hat{q}N + q_n d_{\hat{q}} - \hat{q} d_{q_n} = N$  and, if  $n > 1$ ,  $q_n = q_{n-1} + \hat{q}$ ,  $d_{q_n} = d_{q_{n-1}} + d_{\hat{q}}$  and  $d_{q_{n-1}} + d_{\hat{q}} > N$  for  $q < \hat{q}$ .*

*Proof.* First of all note that all is defined also in the extremal case  $\beta = 0$ . In this case  $\Omega_{\beta, N} = \{1\}$ . Assume first  $n > 1$ . Set  $\tilde{q} = q_n - q_{n-1}$  so that  $d_{q_n} = d_{q_{n-1}} + d_{\tilde{q}}$  since  $d_{q_n} > d_{q_{n-1}}$ . Assume by contradiction that  $\tilde{q} \neq \hat{q}$ . Since  $\tilde{q} < q_n$  we have  $d_{\tilde{q}} < d_{\hat{q}}$ . Let also  $q' = q_n - \hat{q}$  and, as above, we can write  $d_{q_n} = d_{q'} + d_{\hat{q}}$ . Now

$$d_{q_n} - d_{q'} = d_{\hat{q}} < d_{\tilde{q}} = d_{q_n} - d_{q_{n-1}} \implies d_{q_{n-1}} < d_{q'}$$

Since  $q_{n-1} \in \Omega_{\beta, N}$  we must have  $q' > q_{n-1}$ , which is a contradiction because otherwise, being  $q' < q_n$ , we must have  $q' = q_n$ . So  $\tilde{q} = \hat{q}$ . For the last relation note that, since  $q_n$  is the first  $q > q_{n-1}$  such that  $d_q > d_{q_{n-1}}$ , then  $\hat{q}$  is the first such that  $d_{q_{n-1}} + d_{\hat{q}} \leq N$ .

Now consider the first relation. We need to do induction on all the  $\beta$ . So we will write  $d_q^\beta$  and  $q_n^\beta$  in order to remember that those numbers depend on  $\beta$ . The induction statement on  $1 \leq q < N$  is: for any  $0 \leq \beta < N$  and for any  $n$  such that  $q_n^\beta \leq q$  the required formula holds. The base step is  $q = 1$ . In this case we have  $n = 1$ ,  $q_1 = 1$ ,  $\hat{q} = 0$ ,  $d_0 = N$  and the formula can be proven directly. For the induction step we can assume  $q > 1$  and  $n > 1$ . We will write  $\hat{q}_n^\beta$  for the  $\hat{q}$  associated to  $n$  and  $\beta$ . First of all note that, by the relations proved above, we can write

$$\hat{q}_n^\beta N + q_n^\beta d_{q_n^\beta}^\beta - \hat{q}_n^\beta d_{q_n^\beta}^\beta = \hat{q}_n^\beta N + q_{n-1}^\beta d_{q_{n-1}^\beta}^\beta - \hat{q}_n^\beta d_{q_{n-1}^\beta}^\beta$$

and so we have to prove that the second member equals  $N$ . If  $\hat{q}_n^\beta \leq q_{n-1}^\beta$  then  $\hat{q}_{n-1}^\beta = \hat{q}_n^\beta$  and the formula is true by induction on  $q - 1 \geq q_{n-1}^\beta$ . So assume  $\hat{q}_n^\beta > q_{n-1}^\beta$  and set  $\alpha = N - \beta$ . Clearly we will have

$$o = o(\alpha, \mathbb{Z}/N\mathbb{Z}) = o(\beta, \mathbb{Z}/N\mathbb{Z}) \text{ and } d_q^\beta + d_q^\alpha = N \text{ for any } 0 < q < o$$

Moreover

$$d_{q_n^\beta}^\beta < d_q^\beta \text{ for any } 0 < q < q_n^\beta \implies d_{q_n^\beta}^\alpha > d_q^\alpha \text{ for any } 0 < q < \hat{q}_n^\beta \implies \exists l \text{ s.t. } q_l^\alpha = \hat{q}_n^\beta$$

and

$$d_{q_{n-1}^\beta}^\beta \geq d_q^\beta \text{ for any } 0 < q < q_n^\beta \implies d_{q_{n-1}^\beta}^\alpha \leq d_q^\alpha \text{ for any } 0 \leq q < q_l^\alpha = \hat{q}_n^\beta \implies \hat{q}_l^\alpha = q_{n-1}^\beta$$

Using induction on  $q_l^\alpha = \hat{q}_n^\beta < q_n^\beta \leq q$  we can finally write

$$\begin{aligned} N &= \hat{q}_l^\alpha N + q_l^\alpha d_{q_l^\alpha}^\alpha - \hat{q}_l^\alpha d_{q_l^\alpha}^\alpha = q_{n-1}^\beta N + \hat{q}_n^\beta d_{q_{n-1}^\beta}^\alpha - q_{n-1}^\beta d_{q_n^\beta}^\alpha \\ &= q_{n-1}^\beta N + \hat{q}_n^\beta (N - d_{q_{n-1}^\beta}^\beta) - q_{n-1}^\beta (N - d_{q_n^\beta}^\beta) = \hat{q}_n^\beta N + q_{n-1}^\beta d_{q_n^\beta}^\beta - \hat{q}_n^\beta d_{q_{n-1}^\beta}^\beta \end{aligned}$$

□

We continue to keep notation from 4.7. With  $d_q$  we will always mean  $d_q^{N-\alpha}$  as in 4.21. Lemma 4.18 can be restated as:

**Proposition 4.22.** *Let  $A$  be an algebra as in 4.8. Then  $\bar{q}_A \in \Omega_{N-\alpha, N}$ .*

So given an algebra  $A$  as in 4.8 we can associate to it the number  $\bar{q}_A \in \Omega_{N-\alpha, N}$ . Conversely we will see that any  $\bar{q} \in \Omega_{N-\alpha, N}$  admits an algebra  $A$  as in 4.8 such that  $\bar{q} = \bar{q}_A$ . It turns out that all the objects  $z_A, y_A, f_A, \mathcal{E}^A, \delta^A, \hat{q}_A$  and, if  $\lambda_A = 0$ ,  $x_A, w_A$  associated to  $A$  only depend on  $\bar{q}_A$ . Therefore in this subsection, given  $\bar{q} \in \Omega_{N-\alpha, N}$ , we will see how to define such objects independently from an algebra  $A$ .

In this subsection we will consider given an element  $\bar{q} \in \Omega_{N-\alpha, N}$ .

**Definition 4.23.** Set  $\hat{q}$  for the only integer  $0 \leq \hat{q} < \bar{q}$  such that  $d_{\hat{q}} = \min_{0 \leq q < \bar{q}} d_q$ ,  $q' = \bar{q} - \hat{q}$ ,  $z = \bar{q}r$ ,  $y = N - d_{\bar{q}}$ ,

$$x = \begin{cases} N - d_{q'} & \text{if } \bar{q} > 1 \\ N & \text{if } \bar{q} = 1 \end{cases}, \quad w = \begin{cases} q'r & \text{if } \bar{q} > 1 \\ 0 & \text{if } \bar{q} = 1 \end{cases}, \quad f(c) = \begin{cases} x & \text{if } 0 \leq c < \hat{q}r \\ d_{\hat{q}} & \text{if } \hat{q}r \leq c < z \end{cases}$$

We will also write  $\hat{q}_{\bar{q}}, q'_{\bar{q}}, z_{\bar{q}}, x_{\bar{q}}, f_{\bar{q}}, y_{\bar{q}}, w_{\bar{q}}$  if necessary.

*Remark 4.24.* Using notation from 4.21 we have  $\bar{q} = q_n$  for some  $n$  and, if  $n > 1$ , i.e.  $\bar{q} > 1$ ,  $q_{n-1} = q'$ . Note that  $zm = yn, wm = xn, y < x, w < z$ . Moreover, from 4.21 and from a direct computation if  $\bar{q} = 1$ , we obtain  $zx - yw = |M|$ . Finally if  $\bar{q} > 1$  one has relations  $\hat{q}r = z - w$  and  $d_{\hat{q}} = x - y$ .

**Lemma 4.25.** *We have that:*

$$(1) \text{ } f \text{ is a decreasing function and } \sum_{c=0}^{z-1} f(c) = |M|;$$

(2) any element  $t \in M$  can be uniquely written as

$$t = Am + Bn \text{ with } 0 \leq A < z, 0 \leq B < f(A)$$

*Proof.* 1) If  $\bar{q} = 1$  it is enough to note that  $\hat{q} = 0$ ,  $d_0 = N$  and  $Nr = |M|$ . So assume  $\bar{q} > 1$ . We have  $x = N - d_{q'} \geq d_{\hat{q}}$  since  $d_{\bar{q}} = d_{q'} + d_{\hat{q}}$  and

$$\sum_{c=0}^{z-1} f(c) = \hat{q}rx + (\bar{q}r - \hat{q}r)d_{\hat{q}} = (z - w)x + w(x - y) = zx - wy = |M|$$

2) First of all note that the writings of the form  $Am + Bn$  with  $0 \leq A < z$ ,  $0 \leq B < f(A)$  are  $\sum_{c=0}^{z-1} f(c) = |M|$ . So it's enough to prove that they are all distinct. Assume to have writings  $Am + Bn = A'm + B'n$  with  $0 \leq A' \leq A < z, 0 \leq B' < f(A'), 0 \leq B' < f(A')$ .

$A' = B' = 0$ , i.e.  $Am + Bn = 0$ . If  $A = 0$  then  $B = 0$  since  $f(0) = x \leq N$ . If  $A > 0$ , we can write  $A = qr$  for some  $0 < q < \bar{q}$ . In particular  $\bar{q} > 1$  and  $B = d_q < f(A)$ . If  $q < \hat{q}$  then  $f(A) = x = N - d_{q'} > d_q$  contradicting 4.21, while if  $q \geq \hat{q}$  then  $f(A) = d_{\hat{q}} \leq d_q$ .

$A' = B = 0$ , i.e.  $Am = B'n$ . If  $A = 0$  then  $B' = 0$  as above. If  $A > 0$  we can write  $A = qr$  for some  $0 < q < \bar{q}$ . Again  $\bar{q} > 1$ . In particular  $B' = N - d_q < f(0) = x = N - d_{q'}$  and so  $d_{q'} < d_q$ , while  $d_{q'} = \max_{0 < q < \bar{q}} d_q$ .

*General case.* We can write  $(A - A')m + Bn = B'n$  and we can reduce to the previous cases since if  $B \geq B'$  then  $B - B' \leq B < f(A) \leq f(A - A')$ , while if  $B < B'$  then  $B' - B \leq B' < f(A') \leq f(0)$ .  $\square$

**Definition 4.26.** Given  $l \in M$  we set  $(\mathcal{E}_l, \delta_l)$  the unique pair for  $l$  such that  $0 \leq \mathcal{E}_t < z$ ,  $0 \leq \delta_t < f(\mathcal{E}_t)$  and we will consider  $\mathcal{E}, \delta$  as maps  $\mathbb{Z}^M / \langle e_0 \rangle \longrightarrow \mathbb{Z}$ . We will also write  $\mathcal{E}^{\bar{q}}, \delta^{\bar{q}}$  if necessary.

**Proposition 4.27.** Let  $A$  be an algebra as in 4.8. Then

$$z_A = z_{\bar{q}_A}, y_A = y_{\bar{q}_A}, \hat{q}_A = \hat{q}_{\bar{q}_A}, \mathcal{E}^A = \mathcal{E}^{\bar{q}_A}, \delta^A = \delta^{\bar{q}_A}, f_A = f_{\bar{q}_A}$$

and, if  $\lambda_A = 0$ , then  $x_A = x_{\bar{q}_A}$ ,  $w_A = w_{\bar{q}_A}$ .

*Proof.* Set  $\bar{q} = \bar{q}_A$ . Then  $z_A = \bar{q}r = z_{\bar{q}}$  and  $z_A m = y_A n = y_{\bar{q}} n$  implies  $y_A = y_{\bar{q}}$ . Also  $\hat{q}_A = \hat{q}_{\bar{q}}$  by definition. Taking into account 4.17 we can now assume  $\lambda_A = 0$ . We claim that all the remaining equalities follow from  $x_A = x_{\bar{q}}$ . Indeed clearly  $w_A = w_{\bar{q}}$ . Also by definition of  $f_{\bar{q}}$  and thanks to 4.19 we will have  $f_A = f_{\bar{q}}$  and therefore  $\mathcal{E}^A = \mathcal{E}^{\bar{q}}$ ,  $\delta^A = \delta^{\bar{q}}$ , that conclude the proof.

We now show that  $x_A = x_{\bar{q}}$ . If  $\bar{q} = 1$  then  $\hat{q} = 0$  and so, from 4.19, we have  $d_{\hat{q}} = N = x_A = x_1$ . If  $\bar{q} > 1$ , by definition of  $f_{\bar{q}}$  and thanks to 4.25 and 4.19, we can write

$$|M| = \sum_{c=0}^{z_{\bar{q}}-1} f_{\bar{q}}(c) = r\hat{q}_{\bar{q}}x_{\bar{q}} + (z_{\bar{q}} - \hat{q}_{\bar{q}}r)d_{\hat{q}_{\bar{q}}} = \sum_{c=0}^{z_A-1} f_A(c) = r\hat{q}_A x_A + (z_A - \hat{q}_A r)d_{\hat{q}_A}$$

and so  $x_A = x_{\bar{q}}$ .  $\square$

**Definition 4.28.** Define the  $M$ -graded  $\mathbb{Z}[a, b]$ -algebra

$$A^{\bar{q}} = \mathbb{Z}[a, b][s, t] / (s^z - at^y, t^x - bs^w, s^{\hat{q}r} t^{d_{\hat{q}}} - a^{\gamma} b) \text{ where } \gamma = \begin{cases} 0 & \text{if } \bar{q} = 1 \\ 1 & \text{if } \bar{q} > 1 \end{cases}$$

with  $M$ -graduation  $\deg s = m$ ,  $\deg t = n$ . If are given elements  $a_0, b_0$  of a ring  $C$  we will also write  $A_{a_0, b_0}^{\bar{q}} = A^{\bar{q}} \otimes_{\mathbb{Z}[a, b]} C$ , where  $\mathbb{Z}[a, b] \longrightarrow C$  sends  $a, b$  to  $a_0, b_0$ .

**Proposition 4.29.**  $A^{\bar{q}} \in \text{D}(M)\text{-Cov}(\mathbb{Z}[a, b])$ , it is generated in degrees  $m, n$  and  $\{v_l = s^{\mathcal{E}_l} t^{\delta_l}\}_{l \in M}$  is an  $M$ -graded basis for it.



*Proof.* We have to prove that, for any  $l \in M$ ,  $(A^{\bar{q}})_l = \mathbb{Z}[a, b]v_l$  and we can check this over a field  $k$ , i.e. considering  $A = A_{a, b}^{\bar{q}}$  with  $a, b \in k$ . We first consider the case  $a, b \in k^*$ , so that  $s, t \in A^*$ . Let  $\pi: \mathbb{Z}^2 \rightarrow M$  the map such that  $\pi(e_1) = m$ ,  $\pi(e_2) = n$ . The set  $T = \{(a, b) \in \text{Ker } \pi \mid s^a t^b \in k^*\}$  is a subgroup of  $\text{Ker } \pi$  such that  $(z, -y), (-w, x) \in T$ . Since  $\det \begin{pmatrix} z & -w \\ -y & x \end{pmatrix} = zx - wy = |M|$  we can conclude that  $T = \text{Ker } \pi$ . Therefore  $v_l$  generate  $(A^{\bar{q}})_l$  since for any  $c, d \in \mathbb{N}$  we have  $s^c t^d / v_{cm+dn} \in k^*$  and  $0 \neq v_l \in A^*$ .

Now assume  $a = 0$ . If  $\bar{q} = 1$  then  $\hat{q} = w = 0$ ,  $d_{\hat{q}} = x = N$  and so  $A = k[s, t]/(s^z, t^N - b)$  satisfies the requests. If  $\bar{q} > 1$  it is easy to see that  $v_l$  generates  $A_l$ . On the other hand  $\dim_k A = |\{(A, B) \mid 0 \leq A < z, 0 \leq B < x, A \leq \hat{q}r \text{ or } B \leq d_{\hat{q}}\}| = zx - (z - \hat{q}r)(x - d_{\hat{q}}) = zx - yw = |M|$ . The case  $b = 0$  is similar.  $\square$

**Theorem 4.30.** *Let  $k$  be a field. If  $\bar{q} \in \Omega_{N-\alpha, N}$  and  $\lambda \in k$ , with  $\lambda = 0$  if  $\bar{q} = N/(\alpha, N)$ , then*

$$A_{\bar{q}, \lambda} = k[s, t]/(s^{z_{\bar{q}}} - \lambda t^{y_{\bar{q}}}, t^{x_{\bar{q}}}, s^{\hat{q}r} t^{d_{\hat{q}}})$$

is an algebra as in 4.8 with  $\bar{q}_{A_{\bar{q}, \lambda}} = \bar{q}$  and  $\lambda_{A_{\bar{q}, \lambda}} = \lambda$ . Conversely, if  $A$  is an algebra as in 4.8 then  $\bar{q}_A \in \Omega_{N-\alpha, N}$ ,  $\lambda_A \in k$ ,  $\lambda_A = 0$  if  $\bar{q}_A = N/(\alpha, N)$  and  $A \simeq A_{\bar{q}_A, \lambda_A}$ .

*Proof.* Consider  $A = A_{\bar{q}, \lambda}$ , which is just  $A_{\lambda, 0}^{\bar{q}}$ . Clearly  $t \notin A^*$ . On the other hand  $s \notin A^*$  since  $y = 0 \iff z = o(m) \iff \bar{q} = N/(\alpha, N)$ . Therefore  $H_{A/k} = 0$  and  $A$  is an algebra as in 4.8. Moreover clearly  $\bar{q}_A \leq \bar{q}$ . If by contradiction this inequality is strict, we will have a relation  $s^{qr} = \omega t^{y'}$  with  $0 \leq q < \bar{q}$ . Since  $s^{qr} = v_{qrm} \neq 0$  we will have that  $t^{y'} \neq 0$  and  $y' < x$ , a contradiction thanks to 4.25. In particular  $\lambda = \lambda_A$ .

Now let  $A$  be as in 4.8 and set  $\bar{q} = \bar{q}_A$ ,  $\lambda = \lambda_A$ . We already know that  $\bar{q} \in \Omega_{N-\alpha, N}$  (see 4.22). We claim that the map  $A_{\bar{q}, \lambda} \rightarrow A$  sending  $s, t$  to  $v_m, v_n$  is well defined and so an isomorphism. Indeed we have  $v_m^z = \lambda v_n^y$  by definition and, thanks to 4.27, we have  $v_m^{\hat{q}r} v_n^{d_{\hat{q}}} = 0$  since  $d_{\hat{q}} = f_A(\hat{q}r)$  and  $v_n^x = 0$  since  $f_A(0) = x$ . Finally if  $\bar{q} = N/(\alpha, N)$  then  $y = y_A = 0$  and  $z = o(m)$ , so that  $\lambda_A = v_m^{o(m)} = 0$ .  $\square$

**Corollary 4.31.** *If  $k$  is an algebraically closed field then, up to graded isomorphism, the algebras as in 4.8 are exactly  $A_{\bar{q}, 1}$  if  $\bar{q} \in \Omega_{N-\alpha, N} - \{N/(\alpha, N)\}$  and  $A_{\bar{q}, 0}$  if  $\bar{q} \in \Omega_{N-\alpha, N}$ .*

*Proof.* Clearly the algebras above cannot be isomorphic. Conversely if  $\lambda \in k^*$  (and  $\bar{q} < N/(\alpha, N)$ ) the transformation  $t \rightarrow \sqrt[y]{\lambda} t$  with  $y = y_{\bar{q}}$  yields an isomorphism  $A_{\bar{q}, \lambda} \simeq A_{\bar{q}, 1}$ .  $\square$

**4.4. Smooth integral rays for  $h \leq 2$ .** In this subsection we continue to keep notation from 4.7, i.e.  $M = M_{r, \alpha, N}$  and we will considered given an element  $\bar{q} \in \Omega_{N-\alpha, N}$ .

*Remark 4.32.* We have  $z = 1 \iff \bar{q} = r = 1$  and  $x = 1 \iff \bar{q} = N$ . Indeed the first relation is clear, while for the second one note that, by definition of  $x$  and since  $N > 1$ , we have  $x = 1 \iff d_{\hat{q}} = N - 1 \iff \bar{q} = N/(\alpha, N), (\alpha, N) = 1$ .

**Lemma 4.33.** *The vectors of  $K_+$*

$$(4.3) \quad \begin{array}{ll} v_{cm, dn} & 0 < c < z, 0 < d < f(c) \\ v_{m, im} & 0 < i < z - 1 \\ v_{n, jn} & 0 < j < x - 1 \\ v_{m, (z-1)m} & \text{if } z > 1 \\ v_{n, (x-1)n} & \text{if } x > 1 \end{array}$$

form a basis of  $K$ . Assume  $\bar{q}r \neq 1$  and  $\bar{q} \neq N$ , i.e.  $z, x > 1$ , and denote by  $\Lambda, \Delta$  the last two terms of the dual basis of 4.3. Then  $\Lambda, \Delta \in K_+^\vee$  and they form a smooth sequence. Moreover  $\Lambda = 1/|M|(x\mathcal{E} + w\delta)$ ,  $\Delta = 1/|M|(y\mathcal{E} + z\delta)$  and

$$\Lambda_{m,-m} = \Delta_{n,-n} = 1, \quad \Lambda_{n,-n} = \begin{cases} 0 & \text{if } \bar{q} = 1 \\ 1 & \text{otherwise} \end{cases}, \quad \Delta_{m,-m} = \begin{cases} 0 & \text{if } \bar{q} = N/(\alpha, N) \\ 1 & \text{otherwise} \end{cases}$$

*Proof.* Note that we cannot have  $z = x = 1$  since otherwise  $|M| = f(0) = x = 1$ , i.e.  $M = 0$ . The vectors of (4.3) are at most  $\text{rk } K$  since

$$\sum_{c=1}^{z-1} (f(c)-1) + z-2 + x-2 + 2 = \sum_{c=0}^{z-1} (f(c)-1) + z-1 = |M| - z + z - 1 = |M| - 1 = \text{rk } K$$

If  $z = 1$  then (4.3) is  $v_{n,n}, \dots, v_{n,(x-1)n}$ . So  $x = |M| = N$ , i.e.  $n$  generates  $M$ , and (4.3) is a base of  $K$ . In the same way if  $x = 1$ , then  $m$  generates  $M$  and (4.3) is a base of  $K$ .

So we can assume that  $z, x > 1$ .  $\mathcal{E}$  and  $\delta$  define a map  $\mathbb{Z}^M / \langle e_0 \rangle \xrightarrow{(\mathcal{E}, \delta)} \mathbb{Z}^2$ . Denote by  $K'$  the subgroup of  $K$  generated by the vectors in (4.3), except the last two lines. We claim that  $(\mathcal{E}, \delta)|_{K'} = 0$ . This follows by a direct computation just observing that if we have a writing  $Am + Bn$  as in 4.25, 2) then  $(\mathcal{E}, \delta)(e_{Am+Bn}) = (A, B)$ . Consider the diagram

$$\begin{array}{ccccc} & & & \pi & \\ & & & \nearrow & \\ \mathbb{Z}^2 & \xrightarrow{\sigma} & K/K' \hookrightarrow \mathbb{Z}^M / \langle e_0, K' \rangle & \xrightarrow{(\mathcal{E}, \delta)} & \mathbb{Z}^2 \xrightarrow{\tau} \mathbb{Z}^M / \langle e_0, K' \rangle \xrightarrow{p} M \\ & \searrow U & & \nwarrow \tau(e_1)=e_m \tau(e_2)=e_n & \nwarrow p(e_l)=l \end{array}$$

We have  $(\mathcal{E}, \delta)(v_{m,(z-1)m}) = (z, -y)$  since  $y < x = f(0)$  and  $(\mathcal{E}, \delta)(v_{n,(x-1)n}) = (-w, x)$  since  $w < z$ . So  $|\det U| = zx - yw = |M|$  and, since  $\pi \circ U = 0$ ,  $U$  is an isomorphism onto  $\text{Ker } \pi$ . Moreover  $\tau^{-1} = (\mathcal{E}, \delta)$  since  $e_l \equiv \mathcal{E}_l e_m + \delta_l e_n \pmod{K'}$ . It follows that  $\sigma$  is an isomorphism and so (4.3) is a basis of  $K$ .

Consider now the second part of the statement. Clearly  $\Lambda, \Delta \in \langle \mathcal{E}, \delta \rangle_{\mathbb{Q}}$ . Therefore we have

$$\Lambda = a\mathcal{E} + b\delta, \quad \begin{cases} \Lambda(v_{m,(z-1)m}) = 1 = az - yb \\ \Lambda(v_{n,(x-1)n}) = 0 = xb - aw \end{cases} \implies \begin{cases} a = x/|M| \\ b = w/|M| \end{cases}$$

and the analogous relation for  $\Delta$  follows in the same way. Now note that, thanks to 4.30 and 4.27, we have that  $\mathcal{E} = \mathcal{E}^A$ ,  $\delta = \delta^A$  for an algebra  $A$  as in 4.8 with  $\bar{q}_A = \bar{q}$ ,  $\lambda_A = 0$  and sharing the same invariants of  $\bar{q}$ . So we can apply 4.16. We want to prove that  $\Lambda, \Delta \in K_+^\vee$  so that they form a smooth sequence by construction. Assume first that  $\mathcal{E}_{a,b} > 0$ . Clearly  $\Lambda_{a,b}, \Delta_{a,b} \geq 0$  if  $\delta_{a,b} \geq 0$ . On the other hand if  $\delta_{a,b} < 0$  we know that  $\mathcal{E}_{a,b} \geq z$  and  $\delta_{a,b} \geq -y$  and so

$$|M|\Lambda_{a,b} = x\mathcal{E}_{a,b} + w\delta_{a,b} \geq xz - yw = |M| \text{ and } |M|\Delta_{a,b} = y\mathcal{E}_{a,b} + z\delta_{a,b} \geq yz - zy = 0$$

The other cases follows in the same way. It remains to prove the last relations. Since  $-n = \hat{q}rm + (d_{\hat{q}} - 1)n$ , we have  $\mathcal{E}_{n,-n} = \hat{q}r$  and  $\delta_{n,-n} = d_{\hat{q}}$ . Using the relation  $zx - yw = |M|$  the values of  $\Lambda_{n,-n}$ ,  $\Delta_{n,-n}$  can be checked by a direct computation. Similarly, considering the relations  $-m = (\hat{q}r - 1)m + d_{\hat{q}}n$  if  $1 < \bar{q}$ ,  $-m = (r - 1)m + (N - \alpha)n$  if  $\bar{q} = 1$  and  $\alpha \neq 0$ ,  $-m = (r - 1)m$  if  $\alpha = 0$ , we can compute the values of  $\Lambda_{m,-m}$  and  $\Delta_{m,-m}$ .  $\square$

**Proposition 4.34.** *The multiplication of  $A^{\bar{q}}$  (see 4.28) with respect to the basis  $v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$  is:  $a^{\mathcal{E}^\phi}$  if  $\bar{q} = N$ , where  $\phi: M \xrightarrow{\sim} \mathbb{Z}/|M|\mathbb{Z}$ ,  $\phi(m) = 1$ ;  $b^{\mathcal{E}^\eta}$  if  $\bar{q}r = 1$ , where  $\eta: M \xrightarrow{\sim} \mathbb{Z}/|M|\mathbb{Z}$ ,  $\phi(n) = 1$ ;  $a^\Lambda b^\Delta$  if  $\bar{q}r \neq 1$ ,  $\bar{q} \neq N$ , where  $\Lambda, \Delta$  are the rays defined in 4.33.*

*Proof.* In the proof of 4.29 we have seen that if  $x = 1$  ( $\bar{q} = N$ ), then  $M = \langle m \rangle$  and  $A^{\bar{q}} = \mathbb{Z}[a, b][s]/(s^{|M|} - a)$ , while if  $z = 1$  ( $\bar{q}r = 1$ ) then  $M = \langle n \rangle$  and  $A^{\bar{q}} = \mathbb{Z}[a, b][t]/(t^{|M|} - b)$ . So we can assume  $x, z > 1$ . Let  $B$  the  $D(M)$ -cover over  $\mathbb{Z}[a, b]$  given by multiplication  $\psi = a^\Lambda b^\Delta$  and denote by  $\{\omega_l\}_{l \in M}$  a graded basis (inducing  $\psi$ ). By definition of  $\Lambda, \Delta$  we have  $\omega_l = \omega_m^{\varepsilon_l} \omega_n^{\delta_l}$  for any  $l \in M$  and  $\psi_{m, (z-1)m} = a$ ,  $\psi_{n, (x-1)n} = b$ . Therefore

$$\omega_m^z = \omega_m \omega_{(z-1)m} = a \omega_{zm} = a \omega_{yn} = a \omega_n^y, \quad \omega_n^x = \omega_n \omega_{(x-1)n} = b \omega_{xn} = b \omega_{wm} = b \omega_m^w$$

and, checking both cases  $\bar{q} = 1$  and  $\bar{q} > 1$ ,  $\omega_m^{\bar{q}r} \omega_n^{d_{\bar{q}}} = \omega_{-n} \omega_n = a^{\Lambda_{n, \cdot n}} b^{\Delta_{n, \cdot n}} = a^\gamma b$ . In particular we have an isomorphism  $A^{\bar{q}} \rightarrow B$  sending  $v_m, v_n$  to  $\omega_m, \omega_n$ .  $\square$

*Notation 4.35.* From now on  $M$  will be any finite abelian group. If  $\phi: M \rightarrow M_{r, \alpha, N}$  is a surjective map,  $r, \alpha, N$  satisfy the conditions of 4.7,  $\bar{q} \in \Omega_{N-\alpha, N}$  with  $\bar{q}r \neq 1$ ,  $\bar{q} \neq N$  then we set  $\Lambda^{r, \alpha, N, \bar{q}, \phi} = \Lambda \circ \phi_*$ ,  $\Delta^{r, \alpha, N, \bar{q}, \phi} = \Delta \circ \phi_*$ , where  $\Lambda, \Delta$  are the rays defined in 4.33 with respect to  $r, \alpha, N, \bar{q}$ . If  $\phi = \text{id}$  we will omit it.

**Definition 4.36.** Set

$$\Sigma_M = \left\{ (r, \alpha, N, \bar{q}, \phi) \left| \begin{array}{l} 0 \leq \alpha < N, \ r > 0, \ N > 1, \ (r > 1 \text{ or } \alpha > 1) \\ \bar{q} \in \Omega_{N-\alpha, N}, \ \bar{q}r \neq 1, \ \bar{q}\alpha \not\equiv 1 \pmod{N} \\ \bar{q} \neq N/(\alpha, N), \ \phi: M \rightarrow M_{r, \alpha, N} \text{ surjective} \end{array} \right. \right\}$$

and  $\Delta^*: \Sigma_M \rightarrow \{\text{smooth integral extremal rays of } M\}$ .

*Remark 4.37.* Since  $e_2, e_1$  generate  $M_{r, \alpha, N}$ , there exist unique  $r^\vee, \alpha^\vee, N^\vee$  with an isomorphism  $(-)^\vee: M_{r, \alpha, N} \rightarrow M_{r^\vee, \alpha^\vee, N^\vee}$  sending  $e_2, e_1$  to  $e_1, e_2$ . One can check that  $r^\vee = (\alpha, N)$ ,  $N^\vee = rN/(\alpha, N)$  and  $\alpha^\vee = \bar{q}r$ , where  $\bar{q}$  is the only integer  $0 \leq \bar{q} < N/(\alpha, N)$  such that  $\bar{q}\alpha \equiv (\alpha, N) \pmod{N}$ .

If  $A$  is an algebra as in 4.8 for  $M_{r, \alpha, N}$ , then, through  $(-)^\vee$ ,  $A$  can be thought as a  $M_{r^\vee, \alpha^\vee, N^\vee}$ -cover that we will denote by  $A^\vee$ .  $A^\vee$  is an algebra as in 4.8 with respect to  $M_{r^\vee, \alpha^\vee, N^\vee}$  with  $\bar{q}_{A^\vee} = x_A/(\alpha, N)$ ,  $\lambda_{A^\vee} = \mu_A$ . We can define a bijection  $(-)^\vee: \Omega_{N-\alpha, N} - \{N/(N, \alpha)\} \rightarrow \Omega_{N^\vee-\alpha^\vee, N^\vee} - \{N^\vee/(\alpha^\vee, N^\vee)\}$  in the following way. Given  $\bar{q}$  take an algebra  $A$  as in 4.8 for  $M_{r, \alpha, N}$  with  $\bar{q}_A = \bar{q}$  and  $\lambda_A \neq 0$ , which exists thanks to 4.30, and set  $\bar{q}^\vee = \bar{q}_{A^\vee}$ . Taking into account 4.15 and 4.27,  $\bar{q}^\vee = y_{\bar{q}}/(\alpha, N)$  since  $x_A = y_A = y_{\bar{q}}$  and  $(-)^\vee$  is well defined and bijective since  $\lambda_{A^\vee} = \mu_A = \lambda_A^{-1}$ . Note that the condition  $\bar{q}\alpha \equiv 1 \pmod{N}$  is equivalent to  $r^\vee = 1$  and  $\bar{q}^\vee = 1$ .

Finally if  $\phi: M \rightarrow M_{r, \alpha, N}$  is a surjective morphism then we set  $\phi^\vee = (-)^\vee \circ \phi: M \rightarrow M_{r^\vee, \alpha^\vee, N^\vee}$ . Note that in any case we have the relation  $(-)^{\vee\vee} = \text{id}$ . In particular, since  $1^\vee = \alpha/r^\vee$ ,  $\bar{q} = \alpha^\vee/r$  is the dual of  $1 \in \Omega_{N^\vee-\alpha^\vee, N^\vee}$ .

**Proposition 4.38.** Let  $r, \alpha, N$  be as in 4.7,  $\bar{q} \in \Omega_{N-\alpha, N}$  with  $\bar{q}r \neq 1$ ,  $\bar{q} \neq N$  and  $\phi: M \rightarrow M_{r, \alpha, N}$  be a surjective map. Set  $\chi = (r, \alpha, N, \bar{q}, \phi)$ . Then

- (1)  $\bar{q} = N/(\alpha, N)$ :  $\Delta^\chi = \mathcal{E}^\xi$ ,  $\xi: M \xrightarrow{\phi} M_{r, \alpha, N} \rightarrow M_{r, \alpha, N}/\langle m \rangle \simeq \langle n \rangle \simeq \mathbb{Z}/(\alpha, N)\mathbb{Z}$ ;  $\bar{q}\alpha \equiv 1 \pmod{N}$ :  $\Delta^\chi = \mathcal{E}^\zeta$ ,  $\zeta: M \xrightarrow{\phi} M_{r, \alpha, N} = \langle e_1 \rangle$ ;
- (2)  $\bar{q} = 1$ :  $\Lambda^\chi = \mathcal{E}^\omega$ ,  $\omega: M \xrightarrow{\phi} M_{r, \alpha, N} \rightarrow M_{r, \alpha, N}/\langle n \rangle \simeq \langle m \rangle \simeq \mathbb{Z}/r\mathbb{Z}$ ;  
 $w_{\bar{q}} = 1$ :  $\Lambda^\chi = \mathcal{E}^\theta$ ,  $\theta: M \xrightarrow{\phi} M_{r, \alpha, N} = \langle e_2 \rangle$ ;
- (3)  $\bar{q} > 1$  and  $w_{\bar{q}} \neq 1$ :  $\Lambda^\chi = \Delta^{r, \alpha, N, \bar{q}-\bar{q}, \phi}$ .

In particular in the first two cases we have  $h_{\Lambda^\chi} = h_{\Delta^\chi} = 1$ .

*Proof.* We can assume  $M = M_{r, \alpha, N}$  and  $\phi = \text{id}$ . The algebra associated to  $0^{\Lambda^\chi}$ ,  $0^{\Delta^\chi}$  are respectively  $C_{\bar{q}} = k[s, t]/(s^z, t^x - s^w, s^{\bar{q}r} t^{d_{\bar{q}}} - 0^\gamma)$ ,  $B_{\bar{q}} = k[s, t]/(s^z - t^y, t^x, s^{\bar{q}r} t^{d_{\bar{q}}})$  by 4.34.

1) If  $\bar{q} = N/(\alpha, N)$ , then  $z = o(m)$ ,  $y = 0$ ,  $d_{\hat{q}} = (\alpha, N)$  and so  $B_{\bar{q}} = k[s, t]/(s^{o(m)} - 1, t^{(\alpha, N)})$ , the algebra associated to  $0^{\mathcal{E}^\varepsilon}$ . If  $\bar{q}\alpha \equiv 1 \pmod{N}$  then  $r^\vee = (\alpha, N) = 1$  and  $\bar{q} = \alpha^\vee/r$ , i.e.  $\bar{q}^\vee = 1$ . So  $y = 1$  and  $B_{\bar{q}} \simeq k[s]/(s^{|M|})$ , the algebra associated to  $0^{\mathcal{E}^\gamma}$ .

2) If  $\bar{q} = 1$ , then  $z = r$ ,  $\hat{q} = w = 0$ ,  $x = d_{\hat{q}} = N$  and so  $C_1 = k[s, t](t^n - 1, s^r)$ , the algebra associated to  $0^{\mathcal{E}^\omega}$ . If  $w = 1$  then  $\bar{q} > 1$  and so  $C_{\bar{q}} = k[t]/(t^{|M|})$ , the algebra associated to  $0^{\mathcal{E}^\theta}$ .

3) If  $\bar{q} > 1$  then  $H_{C_{\bar{q}}} = 0$  and so  $C_{\bar{q}}$  is an algebra as in 4.8. An easy computation shows that  $z_{C_{\bar{q}}} = w > 1$ , so that  $\bar{q}_{C_{\bar{q}}} = \bar{q} - \hat{q}$  and  $\lambda_{\bar{q}} = 1$ . Therefore  $\Lambda^\chi = \Delta^{r, \alpha, N, \bar{q} - \hat{q}}$  by 4.30.  $\square$

**Proposition 4.39.**  $\Sigma_M^\vee = \Sigma_M$  and we have a bijection

$$\Delta^*: \Sigma_M/(-)^\vee \longrightarrow \{\text{smooth integral extremal rays } \mathcal{E} \text{ with } h_{\mathcal{E}} = 2\}$$

*Proof.*  $\Sigma_M^\vee \subseteq \Sigma_M$  since  $\bar{q}\alpha \not\equiv 1 \pmod{N}$  is equivalent to  $\bar{q}^\vee r^\vee \neq 1$ . Now let  $\mathcal{E}$  be a smooth integral ray such that  $h_{\mathcal{E}} = 2$  and  $A$  the associated algebra over some field  $k$ . We can assume  $H_{A/k} = H_{\mathcal{E}} = 0$ .  $h_{\mathcal{E}} = 2$  means that there exist  $0 \neq m, n \in M$ ,  $m \neq n$  such that  $A$  is generated in degrees  $m, n$ . So  $M = M_{r, \alpha, N}$  as in 4.7 and  $A$  is an algebra as in 4.8. By 4.30 and 4.38 we can conclude that there exist  $\chi \in \Sigma_M$  such that  $\mathcal{E} = \Delta^\chi$ .

Now let  $\chi = (r, \alpha, N, \bar{q}, \phi) \in \Sigma_M$ . We have to prove that  $h_{\Delta^\chi} = 2$  and, since  $M_{r, \alpha, N} \neq 0$ , assume by contradiction that  $h_{\Delta^\chi} = 1$ . We can assume  $M = M_{r, \alpha, N}$  and  $\phi = \text{id}$ . Note that  $h_{\Delta^\chi} = 1$  means that the associated algebra  $B$  is generated in degree  $m$  or  $n$ . If  $A$  is an algebra as in 4.8, then  $A$  is generated in degree  $n$  if and only if  $z = 1$ , that means  $\bar{q}r = 1$ . So  $B$  is generated in degree  $m$ , i.e.  $B^\vee$  is generated in degree  $e_2 \in M_{r^\vee, \alpha^\vee, N^\vee}$ , which is equivalent to  $1 = z_{B^\vee} = \bar{q}^\vee r^\vee = 1$ , and, as we have seen, to  $\bar{q}\alpha \equiv 1 \pmod{N}$ .

Now let  $\chi' = (r', \alpha', N', \bar{q}', \phi') \in \Sigma_M$  such that  $\mathcal{E} = \Delta^\chi = \Delta^{\chi'}$ . Again we can assume  $H_{\mathcal{E}} = 0$  and take  $B, B'$  the algebras associated respectively to  $\chi, \chi'$ . By definition of  $\Delta^*$ ,  $\phi, \phi'$  are isomorphisms. If  $g = \phi' \circ \phi^{-1}: M_{r, \alpha, N} \longrightarrow M_{r', \alpha', N'}$  then we have a graded isomorphism  $p: B \longrightarrow B'$  such that  $p(B_l) = B'_{g(l)}$ . Therefore  $g(\{e_1, e_2\}) = \{e_1, e_2\}$ , i.e.  $g = \text{id}$  or  $g = (-)^\vee$ . It is now easy to show that  $\chi' = \chi$  or  $\chi' = \chi^\vee$ .  $\square$

*Notation 4.40.* We set  $\Phi_M = \{\phi: M \longrightarrow \mathbb{Z}/l\mathbb{Z} \mid l > 1, \phi \text{ surjective}\}$ ,  $\Theta_M^2 = \{\mathcal{E}^\phi\}_{\phi \in \Phi_M} \cup \{(\Lambda^\chi, \Delta^\chi)\}_{\chi \in \Sigma_M}$ , where  $\Sigma_M$  is the set of sequences  $(r, \alpha, N, \bar{q}, \phi)$  where  $r, \alpha, N \in \mathbb{N}$  satisfy  $0 \leq \alpha < N, r > 0, r > 1$  or  $\alpha > 1$ ,  $\bar{q} \in \Omega_{N-\alpha, N}$  satisfy  $\bar{q}r \neq 1$ ,  $\bar{q} \neq N$  and  $\phi: M \longrightarrow M_{r, \alpha, N}$  is a surjective map. Finally set  $\underline{\mathcal{E}} = (\mathcal{E}^\phi, \Delta^\chi)_{\phi \in \Phi_M, \chi \in \Sigma_M/(-)^\vee}$ .

**Theorem 4.41.** *Let  $M$  be a finite abelian group. Then*

$$\{h \leq 2\} = \left( \bigcup_{\phi \in \Phi_M} \mathcal{Z}_M^{\mathcal{E}^\phi} \right) \bigcup \left( \bigcup_{(\Lambda, \Delta) \in \Theta_M^2} \mathcal{Z}_M^{\Lambda, \Delta} \right)$$

*In particular  $\{h \leq 2\} \subseteq \mathcal{Z}_M^{\text{sm}}$ . Moreover  $\pi_{\underline{\mathcal{E}}}: \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \text{D}(M)\text{-Cov}$  induces an equivalence of categories*

$$\left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{\mathcal{Z}}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}} \mid \begin{array}{l} V(z_{\mathcal{E}^1}) \cap \cdots \cap V(z_{\mathcal{E}^r}) \neq \emptyset \text{ iff} \\ r = 1 \text{ or } (r = 2 \text{ and } (\mathcal{E}^1, \mathcal{E}^2) \in \Theta_M^2) \end{array} \right\} = \pi_{\underline{\mathcal{E}}}^{-1}(h \leq 2) \xrightarrow{\sim} \{h \leq 2\}$$

*Proof.* The writing of  $\{h \leq 2\}$  follows from 4.30 and 4.34. Taking into account 4.39, the last part instead follows from 2.45 taking  $\Theta = \Theta_M^2$ .  $\square$

In [Mac03] the authors prove that the toric Hilbert schemes associated to a polynomial algebra in two variables are smooth and irreducible. The same result

is true more generally for multigraded Hilbert schemes, as proved later in [MS10]. Here we obtain an alternative proof in the particular case of equivariant Hilbert schemes:

**Corollary 4.42.** *If  $M$  is a finite abelian group and  $m, n \in M$  then  $M\text{-Hilb}^{m,n}$  is smooth and irreducible.*

*Proof.* Taking into account the diagram in 3.10 it is enough to note that  $D(M)\text{-Cov}^{m,n} \subseteq \{h \leq 2\} \subseteq \mathcal{Z}_M^{\text{sm}}$ .  $\square$

**Proposition 4.43.**  $\Sigma_M = \emptyset$  if and only if  $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$  or  $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$ .

*Proof.* For the only if, note that if  $\phi: M \rightarrow \mathbb{Z}/l\mathbb{Z}$  with  $l > 3$  is surjective, then, taking  $m = l-1$ ,  $n = 1 \in \mathbb{Z}/l\mathbb{Z}$ , we have  $\mathbb{Z}/l\mathbb{Z} \simeq M_{1,l-1,l}$  and  $(1, l-1, l, 2, \phi) \in \Sigma_M$ .

For the converse set  $M = (\mathbb{Z}/p\mathbb{Z})^l$ , where  $p = 2, 3$  and, by contradiction, assume to have  $(r, \alpha, N, \bar{q}, \phi) \in \Sigma_M$ . In particular  $\phi$  is a surjective map  $M \rightarrow M_{r,\alpha,N}$ . If  $e_1, e_2 \in M_{r,\alpha,N}$  are  $\mathbb{F}_p$ -independent then  $M_{r,\alpha,N} = \langle e_1 \rangle \times \langle e_2 \rangle$ ,  $\alpha = 0$ ,  $\Omega_{N-\alpha,N} = \{1\}$  and therefore  $\bar{q} = 1 = N/(\alpha, N)$ , which implies that  $\chi \notin \Sigma_M$ . On the other hand, if  $M_{1,\alpha,p} \simeq \mathbb{Z}/p\mathbb{Z}$ , the only extremal rays for  $\mathbb{Z}/p\mathbb{Z}$  are  $\mathcal{E}^{\text{id}}$  and, if  $p = 3$ ,  $\mathcal{E}^{-\text{id}}$  since  $K_{+\mathbb{Z}/p\mathbb{Z}} \simeq \mathbb{N}^{p-1}$  by 3.18.  $\square$

**Theorem 4.44.** *Let  $M$  be a finite abelian group and  $X$  be a locally noetherian and locally factorial scheme. Set*

$$\mathcal{C}_X^2 = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \begin{array}{l} \text{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \text{if } \nexists \underline{\delta} \in \Theta_M^2 \text{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\}$$

and

$$\mathcal{D}_X^2 = \{Y \xrightarrow{f} X \in D(M)\text{-Cov}(X) \mid h_f(p) \leq 2 \ \forall p \in X \text{ with } \text{codim}_p X \leq 1\}$$

Then  $\pi_{\underline{\mathcal{E}}}$  induces an equivalence of categories

$$\mathcal{D}_X^2 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{C}_X^2) \xrightarrow{\sim} \mathcal{C}_X^2$$

*Proof.* Apply 2.53 with  $\Theta = \Theta_M^2$ .  $\square$

**Remark 4.45.** In general  $\{h \leq 3\}$  doesn't belong to the smooth locus on  $\mathcal{Z}_M$ . For example, if  $M = \mathbb{Z}/4\mathbb{Z}$ ,  $D(M)\text{-Cov} = \{h \leq 3\}$  is integral but not smooth by 3.18 and 3.20.

**4.5. Normal crossing in codimension 1.** In this subsection we want describe, in the spirit of classification 3.42, covers of a locally noetherian and locally factorial scheme with no isolated points and with  $(\text{char } X, |M|) = 1$  whose total space is normal crossing in codimension 1.

**Definition 4.46.** A scheme  $X$  is normal crossing in codimension 1 if for any codimension 1 point  $p \in X$  there exists a local and etale map  $\hat{\mathcal{O}}_{X,p} \rightarrow R$ , where  $R$  is  $k[[x]]$  or  $k[[s, t]]/(st)$  for some field  $k$  and  $\hat{\mathcal{O}}_{X,p}$  denote the completion of  $\mathcal{O}_{X,p}$ .

**Remark 4.47.** If  $X$  is locally of finite type over a perfect field  $k$ , one can show that the above condition is equivalent to having an open subset  $U \subseteq X$  such that  $\text{codim}_X X - U \geq 2$  and there exists an etale coverings  $\{U_i \rightarrow U\}$  with etale maps  $U_i \rightarrow \text{Spec } k[x_1, \dots, x_{n_i}]/(x_1 \cdots x_{r_i})$  for any  $i$ . Anyway we will not use this property.

**Notation 4.48.** In this subsection we will consider a field  $k$  and we will set  $A = k[[s, t]]/(st)$ . Given an element  $\xi \in \text{Aut}_k k[[x]]$  we will write  $\xi_x = \xi(x)$  so that, if  $p \in k[[x]]$  then  $\xi(p)(x) = p(\xi_x)$ . We will call  $I \in \text{Aut}_k k[[s, t]]$  the unique map such that  $I(s) = t$ ,  $I(t) = s$ . Given  $B \in k^*$  we will denote by  $\underline{B}$  the automorphism of  $k[[x]]$  such that  $\underline{B}_x = Bx$ .

Finally, given  $f \in k[[x_1, \dots, x_n]]$  and  $g \in k[x_1, \dots, x_n]$  the notation  $f = g + \dots$  will mean  $f \equiv g \pmod{(x_1, \dots, x_r)^{\deg g + 1}}$ .

The first problem to deal with is to describe the action on  $A$  of a finite group  $M$  and check when  $A$  is a  $D(M)$ -cover over  $A^M$ , assuming to have the  $|M|$ -roots of unity in  $k$ . We start collecting some general facts about  $A$ .

**Proposition 4.49.** *We have:*

- (1)  $A = k \oplus sk[[s]] \oplus tk[[t]]$
- (2) Given  $f, g \in A - \{0\}$  then  $fg = 0$  if and only if  $f \in sk[[s]], g \in tk[[t]]$  or vice versa.
- (3) Any automorphism in  $\text{Aut}_k A$  is of the form  $(\xi, \eta)$  or  $I(\xi, \eta)$  where  $\xi, \eta \in \text{Aut}_k k[[x]]$  and  $(\xi, \eta)(f(s, t)) = f(\xi_s, \eta_t)$ .
- (4) If  $\xi \in \text{Aut}_k k[[x]]$  has finite order then  $\xi = \underline{B}$  where  $B$  is a root of unity in  $k$ . In particular if  $(\xi, \eta) \in \text{Aut}_k A$  has finite order then  $\xi = \underline{B}, \eta = \underline{C}$  where  $B, C$  are roots of unity in  $k$ .
- (5) Let  $f \in k[[x]] - \{0\}$ ,  $B, C$  roots of unity in  $k$ . Then  $f(Bx) = Cf(x)$  if and only if  $C = B^r$  for some  $r > 0$  and, if we choose the minimum  $r$ ,  $f \in x^r k[[x^{o(B)}]]$ .

*Proof.* 1) is straightforward and 2) follows easily writing  $f$  and  $g$  as in 1). For 3) note that if  $\theta \in \text{Aut}_k A$  then  $\theta(s)\theta(t) = 0$  and apply 2). Finally 4) and 5) can be shown looking at the coefficients of  $\xi_x$  and of  $f$ .  $\square$

**Lemma 4.50.** *If  $M < \text{Aut}_k A$  is a finite subgroup containing only automorphisms of the form  $(\xi, \eta)$  then  $A^M \simeq A$ .*

*Proof.* It's easy to show that  $A^M \simeq k[[s^a, t^b]]/(s^a t^b) \simeq A$  where  $a = \text{lcm}\{i \mid \exists (\underline{A}, \underline{B}) \in M \text{ s.t. } \text{ord } A = i\}$  and  $b = \text{lcm}\{i \mid \exists (\underline{A}, \underline{B}) \in M \text{ s.t. } \text{ord } B = i\}$ .  $\square$

Since we are interested in covers of regular in codimension 1 schemes (and  $A$  is clearly not regular) we can focus on subgroups  $M < \text{Aut}_k A$  containing some  $I(\xi, \eta)$ .

**Lemma 4.51.** *Let  $M < \text{Aut}_k A$  be a finite abelian group and assume that  $(\text{char } k, |M|) = 1$  and that there exists  $I(\xi, \eta) \in M$ . Then, up to equivariant automorphisms, we have  $M = \langle I(\text{id}, \underline{B}) \rangle$  or, if  $M$  is not cyclic,  $M = \langle (\underline{C}, \underline{C}) \rangle \times \langle I \rangle$  where  $\underline{B}, \underline{C}$  are roots of unity and  $o(C)$  is even.*

*Proof.* The existence of an element of the form  $I(\xi, \eta)$  in  $M$  implies that  $s$  and  $t$  cannot be homogeneous in  $m_A/m_A^2$ , that  $2 \mid |M|$  and therefore that  $\text{char } k \neq 2$ .

Applying the exact functor  $\text{Hom}_k^M(m_A/m_A^2, -)$ , we get that the surjection  $m_A \rightarrow m_A/m_A^2$  has a  $k$ -linear and  $M$ -equivariant section. This means that there exists  $x, y \in m_A$  such that  $m_A = (x, y)$  and  $M$  acts on  $x, y$  with characters  $\chi, \zeta$ . In this way we get an action of  $M$  on  $k[[X, Y]]$  and an equivariant surjective map  $\phi: k[[X, Y]] \rightarrow A$ . Moreover  $\text{Ker } \phi = (h)$ , where  $h = fg$  and  $f, g \in k[[X, Y]]$  are such that  $\phi(f) = s, \phi(g) = t$ . We can write  $f = aX + bY + \dots, g = cX + dY + \dots$  with  $ad - bc \neq 0$ . Since  $ax + by = s$  in  $m_A/m_A^2$  and  $s$  is not homogeneous there, we have  $a, b \neq 0$ . Similarly we get  $c, d \neq 0$ . In particular, up to normalize  $f, g, x$  we can assume  $b = c = d = 1$ . Now  $h = aX^2 + (a+1)XY + Y^2 + \dots$  and applying Weierstrass preparation theorem [Lan02, Theorem 9.2], there exists a unique  $\tilde{h} \in (h)$  such that  $(\tilde{h}) = (h)$  and  $\tilde{h} = \psi_0(X) + \psi_1(X)Y + Y^2$ . The uniqueness of  $\tilde{h}$  and the  $M$ -invariance of  $(h)$  yield the relations  $m(\tilde{h}) = \eta(m)^2 \tilde{h}$ ,

$$(4.4) \quad m(\psi_0) = \psi_0(\chi(m)X) = \eta(m)^2 \psi_0, \quad m(\psi_1) = \psi_1(\chi(m)X) = \eta(m) \psi_1$$

for any  $m \in M$ . Moreover  $\tilde{h} = \mu h$  where  $\mu \in k[[X, Y]]^*$  and, since the coefficient of  $Y^2$  in both  $h$  and  $\tilde{h}$  is 1, we also have  $\mu(0) = 1$ . In particular  $\psi_0 = aX^2 +$



TABLE 4.1.

$H$	$m, n, r, \alpha, N, \bar{q}$	$B$	$\mathcal{E}$
$\mathbb{Z}/2\mathbb{Z}$	$1, 1, 1, 1, 2, 1$	$\frac{k[[z]][[U]]}{(U^2 - z^2)}$	$2\mathcal{E}^{id}$
$(\mathbb{Z}/2\mathbb{Z})^2$	$(1, 0), (0, 1), 2, 0, 2, 1$	$\frac{k[[z]][[U, V]]}{(U^2 - z, V^2 - z)}$	$\mathcal{E}^{pr_1} + \mathcal{E}^{pr_2}$
$\mathbb{Z}/2l\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $l > 1$	$(1, 0), (1, 1), 2, 2, 2l, 1$	$\frac{k[[z]][[U, V]]}{(U^2 - V^2, V^{2l} - z)}$	$\Delta^{2, 2, 2l, 1}$
$\mathbb{Z}/4l\mathbb{Z}$	$1, 2l + 1, 1, 2l + 1, 4l, 2$	$\frac{k[[z]][[U, V]]}{(U^2 - V^2, V^{2l+1} - zU, UV^{2l-1} - z)}$	$\Delta^{1, 2l+1, 4l, 2}$
$\mathbb{Z}/2l\mathbb{Z}$ $l > 1$ odd	$1, l + 1, 2, 2, l, 1$	$\frac{k[[z]][[U, V]]}{(U^2 - V^2, V^l - z)}$	$\Delta^{2, 2, l, 1}$

$\dots$  and  $\psi_1 = (a + 1)X + \dots$  and so  $(a + 1)(\chi - \zeta) = 0$  by 4.4. Since  $s$  is not homogeneous in  $m_A/m_A^2$ ,  $\chi \neq \eta$  and  $a = -1$ . Since  $\text{char } k \neq 2$  we can write  $\tilde{h} = (Y + \psi_1/2)^2 - (\psi_1^2/4 - \psi_0) = y'^2 - z'$ . Note that  $y', z'$  are homogeneous thanks to 4.4. Moreover, by Hensel's lemma, we can write  $z' = X^2 + \dots = X^2 q^2$  for an homogeneous  $q \in k[[x]]$  with  $q(0) = 1$ . So  $x' = xq$  is homogeneous and  $\tilde{h} = y'^2 - x'^2$ . This means that we can assume  $s = x - y$ ,  $t = x + y$ . In particular  $\chi^2 = \eta^2$  and  $M$  acts on  $s, t$  as

$$m(s) = \frac{\chi + \zeta}{2}(m)s + \frac{\chi - \zeta}{2}(m)t \quad m(t) = \frac{\chi - \zeta}{2}(m)s + \frac{\chi + \zeta}{2}(m)t$$

Consider the exact sequence

$$(4.5) \quad 0 \longrightarrow H \longrightarrow M \xrightarrow{\chi/\eta} \{-1, 1\} \longrightarrow 0$$

If  $M$  is cyclic, say  $M = \langle m \rangle$ , we have  $\chi(m) = -\eta(m)$  and so  $m = I(\underline{B}, \underline{B})$ , where  $B = (\chi(m) - \eta(m))/2$  is a root of unity. Up to normalize  $s$  we can write  $m = I(\text{id}, \underline{B})$ .

Now assume that  $M$  is not cyclic.  $H$  acts on  $s$  and  $t$  with the character  $\chi|_H = \zeta|_H$  and this yields an injective homomorphism  $\chi|_H: H \longrightarrow \{\text{roots of unity of } k\}$ . So  $H = \langle (\underline{C}, \underline{C}) \rangle$  for some root of unity  $C$ . The extension 4.5 corresponds to an element of  $\text{Ext}^1(\mathbb{Z}/2\mathbb{Z}, H) \simeq H/2H$  that differs to the sequence  $0 \longrightarrow H \longrightarrow \mathbb{Z}/2o(C)\mathbb{Z} \longrightarrow \{-1, 1\} \longrightarrow 0$ . So  $H/2H \simeq \mathbb{Z}/2\mathbb{Z}$ ,  $o(C)$  is even and the sequence 4.5 splits. We can conclude that  $M = \langle (\underline{C}, \underline{C}) \rangle \times \langle m \rangle$ , where  $m = I(\underline{D}, \underline{D})$  for some root of unity  $D$  and  $o(m) = 2$ . Normalizing  $s$  we can write  $m = I(\text{id}, \underline{D}) = I$ .  $\square$

**Proposition 4.52.** *Let  $M < \text{Aut}_k A$  be a finite abelian group such that  $(\text{char } k, |M|) = 1$  and that there exists  $I(\xi, \eta) \in M$ . Also assume that  $k$  contains the  $|M|$ -roots of unity. Then  $A^M \simeq k[[z]]$ ,  $A \in \text{D}(M)\text{-Cov}(A^M)$  and only the following possibilities happen: there exists a row of table 4.1 such that  $M \simeq H$  is generated by  $m, n$ ,  $H \simeq M_{r, \alpha, N}$ ,  $A \simeq B$  as  $M$ -covers, where  $\deg U = m$ ,  $\deg V = n$  and  $A$  over  $A^M$  is given by multiplication  $z^{\mathcal{E}}$ . Moreover all the rays of the form  $\Delta^*$  in the table satisfy  $h_{\Delta^*} = 2$ .*

*Proof.* We can reduce to the actions obtained in 4.51. We first consider the cyclic case, i.e.  $M = \langle I(\text{id}, \underline{B}) \rangle \simeq \mathbb{Z}/2l\mathbb{Z}$  where  $l = o(B)$ . There exists  $E$  such that  $E^2 = B$ . Given  $0 \leq r < |M| = 2l$ , we want to compute  $A_r = \{a \in A \mid I(\text{id}, \underline{B})a = E^r a\}$ .  $a = c + f(s) + g(t) \in A_r$  if and only if  $a = 0$  if  $r > 0$ ,  $f(t) = E^r g(t)$  and  $g(Bs) = E^r f(s)$ . Moreover  $f(t) = E^{-r} g(Bt) = E^{-2r} f(Bt) \implies f(Bt) = E^r f(t)$ . If we denote by  $\delta_r$  the only integer such that  $0 \leq \delta_r < l$  and  $\delta_r \equiv r \pmod{l}$ , we have that, up to constants,  $A^r$  is given by elements of the form  $E^r f(s) + f(t)$  for  $f \in X^{\delta_r} k[[X^l]]$ . Call  $\beta = s^l + t^l \in A_0 = A^M$  and  $v_r = E^r s^{\delta_r} + t^{\delta_r}$ ,  $v_0 = 1$ . We

claim that  $A^M = A_0 = k[[\beta]]$  and  $v_r$  freely generates  $A_r$  as an  $A_0$  module. The first equality holds since  $A_0$  is a domain and we have relations

$$\sum_{n \geq 1} a_n s^{nl} + \sum_{n \geq 1} a_n t^{nl} = \sum_{n \geq 1} a_n (s^l + t^l)^n = \sum_{n \geq 1} a_n \beta^n$$

while the second claim come from the relation

$$E^r s^{\delta_r}(c + h(s)) + t^{\delta_r}(c + h(t)) = (E^r s^{\delta_r} + t^{\delta_r})(c + h(s) + h(t)) \text{ for } h \in X^l k[[X^l]]$$

and the fact that  $v_r$  is not a zero divisor in  $A$ .

So  $A \in \mathcal{D}(M)\text{-Cov}(k[[\beta]])$  and it is generated by  $v_1 = Es + t$  and  $v_{l+1} = -Es + t$  and so in degrees 1 and  $l + 1$ . If  $l = 1$ , so that  $M \simeq \mathbb{Z}/2\mathbb{Z}$ ,  $B = 1$ ,  $E = -1$  and  $v_1^2 = \beta^2$ . This means that  $A \simeq k[[\beta]][U]/(U^2 - \beta^2)$  and its multiplication over  $k[[\beta]]$  is given by  $\beta^{2\mathcal{E}^{\text{id}}}$ . This is the first row. Assume  $l > 1$  and set  $m = 1$ ,  $n = l + 1$ . Note that  $0 \neq m \neq n$  and that  $M \simeq M_{r,\alpha,N}$  for some  $r$ ,  $\alpha$ ,  $N$  that we are going to compute.

$l$  odd. We have  $r = \alpha = 2$  and  $N = l$  since  $\langle l + 1 \rangle = \langle 2 \rangle \subseteq \mathbb{Z}/2l\mathbb{Z}$ . Consider  $\bar{q} = 1 \in \Omega_{N,N-\alpha}$  and the associated numbers are  $z = r = 2$ ,  $y = \alpha = 2$ ,  $\hat{q} = 0$ ,  $d_{\hat{q}} = x = N = l$ ,  $w = 0$ . Since  $v_1^z = v_{l+1}^y$  and  $v_{l+1}^l = \beta$ , we will have  $A \simeq_{k[[\beta]]} A_{\lambda,\mu}^1$  where  $\lambda, \mu = 1, \beta \in k[[\beta]]$  (see 4.28) and therefore the multiplication is  $\beta^{\Delta^{2,2,l,1}}$  by 4.34. This is the fifth row.

$l$  even. We have  $r = 1$ ,  $\alpha = l + 1$ ,  $N = 2l$  since  $\langle l + 1 \rangle = \mathbb{Z}/2l\mathbb{Z}$ . Since  $d_1 = l - 1 \equiv -\alpha$  and  $d_2 = 2l - 2 \equiv 2(-\alpha)$  modulo  $2l$  we can consider  $\bar{q} = 2 \in \Omega_{N-\alpha,N}$ . The associated numbers are  $z = y = 2$ ,  $\hat{q} = 1$ ,  $d_{\hat{q}} = l - 1$ ,  $x = N - (d_{\bar{q}} - d_{\hat{q}}) = l + 1$ ,  $w = 1 \equiv xn = (l + 1)^2 \pmod{2l}$ . Since  $v_1^z = v_{l+1}^y$ ,  $v_{l+1}^x = \beta v_1$  and  $v_1^{\hat{q}} v_{l+1}^{d_{\hat{q}}} = \beta$ , we will have  $A \simeq_{k[[\beta]]} A_{\lambda,\mu}^2$  where  $\lambda, \mu = 1, \beta \in k[[\beta]]$  whose multiplication is  $\beta^{\Delta_{1,l+1,2l,2}}$ . This is the fourth row.

Now consider the case  $M = \langle \underline{C}, \underline{C} \rangle \times \langle I \rangle$  with  $o(C) = l$  even. Set  $\beta = s^l + t^l$ ,  $v_{1,0} = s + t$  and  $v_{1,1} = -s + t$ . Note that  $v_{r,i}$  is homogeneous of degree  $(r, i)$ . Set  $m = (1, 0)$ ,  $n = (1, 1)$ . They are generators of  $M$  and so  $M \simeq M_{r,\alpha,N}$  for some  $r, \alpha, N$ . We have  $N = o(n) = l$ ,  $r > 1$  since  $\langle n \rangle \neq M$  and so  $r = 2$  since  $2m = 2n$ . If  $l = 2$  we get  $\alpha = 0$  and if  $l > 2$  we get  $\alpha = 2$ . Choose  $\bar{q} = 1$  so that the associated numbers are  $z = 2$ ,  $y = \alpha$ ,  $\hat{q} = 0$ ,  $d_{\hat{q}} = x = N = l$ ,  $w = 0$ . As done above, it is easy to see that  $A^M = k[[\beta]]$ . We first consider the case  $l = 2$ . Since  $v_{1,0}^2 = \beta$ ,  $v_{1,1}^2 = \beta$  we get a surjection  $A_{\beta,\beta}^1 \rightarrow A$  which is an isomorphism by dimension. From the writing of  $A_{\beta,\beta}^1$  we can deduce directly that the multiplication is  $\beta^{\mathcal{E}^{\text{pr}_1} + \mathcal{E}^{\text{pr}_2}}$ , where  $\text{pr}_i: (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \mathbb{Z}/2\mathbb{Z}$  are the two projections. This is the second row.

Now assume  $l > 2$ . Since  $v_{1,0}^2 = v_{1,1}^2$  and  $v_{1,1}^l = \beta$  and arguing as above we get  $A \simeq_{k[[\beta]]} A_{\lambda,\mu}^1$  where  $\lambda, \mu = 1, \beta \in k[[\beta]]$  and the multiplication is  $\beta^{\Delta^{2,2,l,1}}$ . This is the third row.

Finally the last sentence is clear by definition of  $\Sigma_M$  and 4.39.  $\square$

*Remark 4.53.* If  $X$  is a locally noetherian integral scheme and there exists a  $\mathcal{D}(M)$ -cover  $Y/X$  such that  $Y$  is normal crossing in codimension 1, then  $X$  is defined over a field. Indeed if  $\text{char } \mathcal{O}_X(X) = p$  then  $\mathbb{F}_p \subseteq \mathcal{O}_X(X)$ . Otherwise  $\mathbb{Z} \subseteq \mathcal{O}_X(X)$  and we have to prove that any prime number  $q \in \mathbb{Z}$  is invertible. We can assume  $X = \text{Spec } R$ , where  $R$  is a local noetherian domain. If  $\dim R = 0$  then  $R$  is a field, otherwise, since  $\text{ht}(q) \leq 1$ , we can assume  $\dim R = 1$  and  $R$  complete. By definition of normal crossing in codimension 1, if  $Y = \text{Spec } S$  and  $p \in Y$  is over  $m_R$  we have a flat and local map  $R \rightarrow S \rightarrow S_p \rightarrow B$ , such that  $B$  contains a field  $k$ .  $q$  is a non zero divisor in  $R$  and therefore in  $B$ . In particular  $0 \neq q \in k^* \subseteq B^*$  and  $q \in R^*$ .



**Theorem 4.54.** *Let  $M$  be a finite abelian group,  $X$  be a locally noetherian and locally factorial scheme with no isolated points and  $(\text{char } X, |M|) = 1$ . Define*

$$NC_X^1 = \{Y/X \in \mathcal{D}(M)\text{-Cov}(X) \mid Y \text{ is normal crossing in codimension 1}\}$$

*Then  $NC_X^1 \neq \emptyset$  if and only if each connected component of  $X$  is defined over a field. In this case define*

$$\underline{\mathcal{E}} = \left( \begin{array}{l} \mathcal{E}^\phi \text{ for } \phi: M \rightarrow \mathbb{Z}/l\mathbb{Z} \text{ surjective with } l \geq 1, \\ \Delta^{2,2,l,1,\phi} \text{ for } \phi: M \rightarrow M_{2,2,l} \text{ surjective with } l \geq 3, \\ \Delta^{1,2l+1,4l,2,\phi} \text{ for } \phi: M \rightarrow M_{1,2l+1,4l} \text{ surjective with } l \geq 1 \end{array} \right)$$

and  $\mathcal{C}_{NC,X}^1$  as the subcategory of  $\mathcal{F}_{\underline{\mathcal{E}}}(X)$  of objects  $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$  such that:

- (1) for all  $\mathcal{E} \neq \delta \in \underline{\mathcal{E}}$ ,  $\text{codim } V(z_{\mathcal{E}}) \cap V(z_{\delta}) \geq 2$  except the case where  $\mathcal{E} = \mathcal{E}^\phi$ ,  $\delta = \mathcal{E}^\psi$

$$\begin{array}{ccc} & \phi & \\ & \curvearrowright & \\ M & \twoheadrightarrow & (\mathbb{Z}/2\mathbb{Z})^2 \\ & \curvearrowleft & \\ & \psi & \end{array} \quad \begin{array}{c} \text{pr}_1 \\ \searrow \\ \text{pr}_2 \end{array} \quad \begin{array}{c} \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \end{array}$$

in which  $v_p(z_{\mathcal{E}^\phi}) = v_p(z_{\mathcal{E}^\psi}) = 1$  if  $p \in Y^{(1)} \cap V(z_{\mathcal{E}^\phi}) \cap V(z_{\mathcal{E}^\psi})$ ;

- (2) for all  $\mathcal{E} \in \underline{\mathcal{E}}$  and  $p \in Y^{(1)}$   $v_p(z_{\mathcal{E}}) \leq 2$  and  $v_p(z_{\mathcal{E}}) = 2$  if and only if  $\mathcal{E} = \mathcal{E}^\phi$  where  $\phi: M \rightarrow \mathbb{Z}/2\mathbb{Z}$  is surjective.

Then we have an equivalence of categories

$$\mathcal{C}_{NC,X}^1 = \pi_{\underline{\mathcal{E}}}^{-1}(NC_X^1) \xrightarrow{\sim} NC_X^1$$

*Proof.* The first claim comes from 4.53. We will make use of 4.44. If  $Y/X \in NC_Y^1$  and  $p \in Y^{(1)}$  we have  $h_{Y/X}(p) \leq \dim_{k(p)} m_p/m_p^2 \leq 2$  since etale maps preserve tangent spaces and  $\dim m_A/m_A^2 \leq 2$ . So  $NC_X^1 \subseteq \mathcal{D}_X^2$ .

Let  $\underline{\delta}$  be the sequence of smooth integral rays given in 4.44. We know that  $\pi_{\underline{\delta}}^{-1}(NC_X^1) \subseteq \mathcal{C}_X^2$ . So we have only to prove that  $\pi_{\underline{\mathcal{E}}}^{-1}(NC_X^1) \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X) \subseteq \mathcal{F}_{\underline{\delta}}(X)$  and that any element  $Y \in NC_X^1$  locally satisfies the requests of the theorem. So we can reduce to the case where  $X = \text{Spec } R$ , where  $R$  is a complete DVR. Since  $R$  contains a field, then  $R \simeq k[[x]]$ . Let  $\chi \in \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}_X^2)$  and  $D$  the associated  $M$ -cover over  $R$ . Let  $C$  be the maximal torsor of  $D/R$  and  $H = H_{D/R}$ . Note that, for any maximal ideal  $q$  of  $C$  we have  $C_q \simeq k(q)[[x]]$  since  $C/R$  is etale. Moreover  $\text{Spec } D \in NC_X^1$  for  $M$  if and only if for any maximal prime  $p$  of  $D$   $\text{Spec } D_p \in NC_{\text{Spec } C_q}^1$  for  $M/H$ , where  $q = C \cap p$ . In the same way  $\chi \in \mathcal{C}_{NC,X}^1$  for  $M$  if and only if, for any maximal prime  $q$  of  $C$ ,  $\chi|_{\text{Spec } C_q} \in \mathcal{C}_{NC, \text{Spec } C_q}^1$  for  $M/H$ . We can therefore reduce to the case  $H_{D/R} = 0$ . We can also assume that  $k$  contains the  $|M|$ -roots of unity.

First assume that  $\text{Spec } D \in NC_Y^1$ . If  $D$  is regular, the conclusion comes from 3.42. So assume  $D$  not regular and denote by  $\mu: R = k[[x]] \rightarrow D$  the associated map. We know that  $D/m_A = k$ . By Cohen's structure theorem we can write  $D = k[[y]]/I$  in such a way that  $\mu|_k = \text{id}_k$ . By definition, since  $D$  is local and complete, there exists an etale extension  $D \rightarrow B = L[[s, t]]/(st)$ . Using the properties of complete rings,  $B/D$  is finite and so  $B \simeq D \otimes_k L$ . Up to change the base  $R$  with  $R \otimes_k L$  we can assume that  $D \simeq k[[s, t]]/(st)$ .  $\mu|_k: k \rightarrow D$  extends to a map  $\nu: D \rightarrow D$  sending  $s, t$  to themselves. This map is clearly surjective. Since  $\text{Spec } D$  contains 3 points,  $\nu$  induces a closed immersion  $\text{Spec } D \rightarrow \text{Spec } D$  which is a bijection. Since  $D$  is reduced  $\nu$  is an isomorphism. This shows that we can write  $D = A = k[[s, t]]/(st)$  in such a way that  $\mu|_k = \text{id}_k$ . So  $\mathcal{D}(M) \simeq \underline{M}$  acts as a subgroup of  $\text{Aut}_k A$  such that  $A^M \simeq k[[z]]$ . In particular, by 4.50, there exists

$I(\xi, \eta) \in M$ . Up to equivariant isomorphisms the possibilities allowed are described in 4.52 and coincides with the ones of the statement. So  $\chi \in \mathcal{C}_{NC,X}^1$ .

Now assume that  $\chi \in \mathcal{C}_{NC,X}^1$ . By definition of  $\pi_{\underline{\mathcal{E}}}$  the multiplication that defines  $D$  over  $R$  is something of the form  $\psi = \lambda z^{\mathcal{E}}$ , where  $\lambda$  is an  $M$ -torsor and  $\mathcal{E}$  is one of the ray of table 4.1. The case  $\mathcal{E} = \mathcal{E}^{\phi}$  comes from 3.42. Since, in our hypothesis, an  $M$ -torsor (in the fppf meaning) is also an etale torsor, up to change the base  $R$  by an etale neighborhood (that maintains the form  $k[[x]]$ ), we can assume  $\lambda = 1$ . In this case, thanks to 4.51 and 4.52, we can conclude that  $A \simeq k[[s, t]]/(st)$  as required.  $\square$

**Corollary 4.55.** *Let  $X$  be a locally noetherian and regular in codimension 1 (normal) scheme with no isolated points,  $M$  be a finite abelian group with  $(\text{char } X, |M|) = 1$  and  $|M|$  odd. If  $Y/X$  is a  $D(M)$ -cover and  $Y$  is normal crossing in codimension 1 then  $Y$  is regular in codimension 1 (normal).*

*Proof.* Since  $Y/X$  has Cohen-Macaulay fibers it is enough to prove that  $Y$  is regular in codimension 1 by Serre's criterion. So we can assume  $X = \text{Spec } R$ , where  $R$  is a DVR, and apply 3.42 just observing that  $\widehat{\text{Reg}}_X^1 = \mathcal{C}_{NC,X}^1$ .  $\square$

*Remark 4.56.* We keep notation from 4.54 and set  $\underline{\delta} = (\mathcal{E}^{\eta}, \eta: M \rightarrow \mathbb{Z}/d\mathbb{Z} \text{ surjective}, d > 1)$ . We have that  $\pi_{\underline{\delta}}^{-1}(NC_X^1) = \mathcal{C}_{NC,X}^1 \cap \mathcal{F}_{\underline{\delta}}$ , i.e. the covers  $Y/X \in NC_X^1$  writable only with the rays in  $\underline{\delta}$ , has the same writing of  $\mathcal{C}_{NC,X}^1$  but with object in  $\mathcal{F}_{\underline{\delta}}$ . Therefore the multiplications that yield a not smooth but with normal crossing in codimension 1 covers are only  $\mathcal{E}^{\phi} + \mathcal{E}^{\psi}$ , where  $\phi, \psi$  are morphism as in 1), and  $\mathcal{E}^{2\phi}$ , where  $\phi: M \rightarrow \mathbb{Z}/2\mathbb{Z}$  is surjective. This result can also be found in [AP11, Theorem 1.9]. In particular, if  $M = (\mathbb{Z}/2\mathbb{Z})^r$ , where  $\underline{\delta} = \underline{\mathcal{E}}$  thanks to 4.43, these are the only possibilities.

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